Characterisation of the Quantum Fuzzy Measure Based on a Quantum Circuit

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***Abstract*—The fuzzy measure (FM) is a powerful means to capture the worths of combinations of sub-sources, offering potential for applications from sensor fusion to ensemble classification. In most approaches, worths are treated as numeric values, which may not reflect real-world settings, as they may contain underlying uncertainty. Intervals and fuzzy sets have been explored to capture the uncertainty in the worths. Building on these ideas, this paper leverages quantum computing, specifically qubits, to interpret the uncertainty in the worths. The approach puts forward a new way of understanding the FM in the sense of quantum mechanics. We define the boundary conditions and the monotonicity constraint, and articulate that a measurement of a given quantum FM (QFM) can be interpreted as a binary FM (BFM). Compared to a BFM, which encodes a single 0/1 configuration of worths, a QFM captures information of all configurations: measuring it can yield any configuration, each occurring with a certain probability. A quantum circuit is proposed to ensure that the measurement of a QFM, i.e. a BFM, always follows the monotonicity of the numeric FM. Going one step further, we apply the Choquet fuzzy integral (CFI) based on a QFM and show that it can be treated as the probability-weighted sum, i.e. the expectation, of the CFI over all possible measurement outcomes of this QFM. We conclude by discussing limitations and charting the steps for future work on real-world applications of the QFM, in particular for ensemble classifiers.**

***Index Terms*—Fuzzy measure, quantum computing, fuzzy integral, information aggregation**

I. Introduction

The fuzzy measure (FM) [1] is a powerful tool to represent the worths (weights) of subsets of information sources, such as different experts in an expert decision-making context [2]; the worths of different sensors, in a sensor fusion context [3]; or the worths of various classifiers, in an ensemble classification context [4].

Typically, the worths in a FM are numeric values between 0 and 1. However, the worths themselves may contain underlying uncertainty, which can be captured by intervals or fuzzy sets. This leads to research on exploring the interpretation of FM based on intervals or fuzzy sets, resulting in interval-valued or fuzzy set-valued FMs [5].

Quantum computing is based on the rules of quantum mechanics, where the basic information unit is a qubit. Unlike a classical bit, two coefficients are assigned to represent the state amplitudes of a qubit. The only way to extract information from a qubit is by making a measurement. Without measurement, the qubit remains in a superposition state, leading to a layer of uncertainty. In addition, qubits are treated as the weights in the Quantum-Inspired Neural Network [6]–[8], which shows that qubits can be a representation of weights.

Building on the above, this paper proposes a FM interpretation based on quantum computing, which uses qubits to capture the worths within the FM. We refer to this type of FM as the quantum FM (QFM). In this case, it can be interpreted that the state of the worths ($|0\rangle$ or $|1\rangle$) in the QFM is uncertain unless we make a measurement on it.

A QFM is highly related to a binary FM (BFM) [9]. As discussed in Section II-A1, the worths in a BFM take values in $\{0, 1\}$: a value of 0 indicates that the combination of sources corresponding to this worth is insufficient to be considered, while a value of 1 indicates that it is worth considering. In this sense, the worths in a QFM can be interpreted as being in an uncertain state: whether the corresponding combinations of sources are sufficient to be considered or not is only revealed upon measurement. Moreover, a measurement of a QFM is also interpreted as a BFM. Unlike a BFM, which represents a single 0/1 configuration, a QFM can capture all 0/1 configurations: measuring it can yield any possible configuration, i.e. BFMs, with a certain probability.

We define the boundary conditions and the monotonicity constraint of the QFM. However, if we make a measurement on the QFM, the resulting QFM may not follow the monotonicity constraint of the numeric FM. To address this, a quantum gate, discussed in Section III-B, is proposed as the monotonicity constraint gate (MC), which is a five-dimensional quantum transformation with four control qubits and one target qubit. Then, a quantum circuit, composed of y-axis rotation gates (RY gate as discussed in Section II-C1) and MC gates, is developed in Section III-B to make sure that one measurement of the QFM follows the monotonicity and to parametrize the QFM.

We refer to the circuit as Q and analyse the measurement of it. The measurement acts as a bit-string and can be transformed into a BFM, which corresponds to a possible measurement of a QFM. We use Qiskit [10] to estimate the probability of each possible measurement outcome by simulating the measurement multiple times and counting how many times each possible measurement outcome occurs. We use a synthetic example to illustrate the counts of each possible measurement outcome and present the interpretation of the QFM based on the measurement of Q.

Furthermore, as discussed in [5], if we fuse information using the Choquet fuzzy integral (CFI) based on interval-valued or fuzzy set-valued FM, the output of the FI is an interval or a fuzzy set, respectively. In reference to this, this paper uses the CFI based on the QFM to fuse information, and the output is expected to be similar to the qubits. We further articulate that the CFI can be interpreted as the weighted sum of the CFI over all possible measurement outcomes of the QFM, where the weights are the probabilities of the corresponding measurement outcomes. This probability-weighted sum can also be interpreted as the expectation of the CFI based on the QFM in one measurement.

The remainder of the paper is organized as follows. Section II provides the background knowledge of this paper including a brief introduction to the FM, the FI, and quantum computing. Section III provides details on the QFM. In Section IV, we explore the CFI regarding the QFM. Finally, Section V concludes the paper and presents future work.

II. BACKGROUND

A. Fuzzy Measure

The FM [1] is an important method to model the 'worth' of subsets of information sources. Let $S = \{s_1, s_2, \cdots, s_I\}$ be a nonempty finite set where $s_i \in S$, $i = \{1, \cdots, I\}$ represent the i^{th} information sources. The FM g is a set function which maps the power set of S to the given weights, i.e. $g : 2^S \to [0, 1]$, and follows these constraints:

1) Boundary conditions:
   $g(\varnothing) = 0$ and $g(S) = 1$
2) Monotonicity:
   If $A \subseteq B \subseteq S$, then $g(A) \leq g(B)$

1) Binary FM: If the worths in a FM take values only in $\{0, 1\}$, it is referred to as a binary FM (BFM) [9]. Let g^* be a BFM, and let $g(\{s_1\}) \to$, $g(\{s_1, s_2\}) \to$, $g(\{s_1, s_2, s_3\}) \to g(\{s_1, s_2, s_3, s_4\})$ be a path of the worths. If we know the first worths of 1 in this path is $g(\{s_1, s_2\})$, the others worths can be automatically deduced as $g(\{s_1\}) = 0$, $g(\{s_1, s_2, s_3\}) = 1$ and $g(\{s_1, s_2, s_3, s_4\}) = 1$. This property of BFM leads to a simpler representation and more efficient computation [11].

One way to think about the BFM is as follows. Each worth in the BFM that is equal to 0, indicates that the corresponding combination of sources is insufficient to be considered, and more sources are needed. However, worths equal to 1 indicates that the corresponding combinations of sources are worth considering. In addition, the BFM has been applied in a variety of areas including, classifier fusion [12], multi-resolution fusion [13] and model identification [14].

B. Fuzzy Integral

The Choquet fuzzy integral (CFI) [15] is one of the most common FI methods, fusing evidence in respect to the given FM g. Let $h : S \to R$ be the integrand, i.e. the evidence, they are defined as follows:

$$\int_C h \circ g = C_g(h) = \sum_{i=1}^{I} h(s_{(i)})(g(A_{(i)}) - g(A_{(i-1)})), \quad (1)$$

where $S = \{s_1, \cdots, s_I\}$ has been sorted such that $h(s_{(1)}) \geq h(s_{(2)}) \geq \cdots \geq h(s_{(I)})$, $A_{(i)} = \{s_{(1)}, \cdots, s_{(i)}\}$ and $g(A_{(0)}) = 0$. This paper focuses on the CFI based on the BFM, which can be interpreted as looking for the first $A_{(i)}$, i.e. smallest i, relative to the h sort, that has non-zero worth [9].

C. Quantum Computing

The qubit is the basic information unit in quantum computing, being defined by a unitary and bi-dimensional state vector. Following the notation in [16], a qubit can be denoted as

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle\,, \quad (2)$$

where the coefficients α and β are two complex numbers and $|\alpha|^2 + |\beta|^2 = 1$.

1) Quantum circuit: The quantum circuit provides a quantum model based on the composition of n-dimensional quantum gates. Here, we introduce some frequently used quantum gates together with their circuit representations.

The NOT gate (X) can transform a qubit $|\psi\rangle$ as

$$X\,|\psi\rangle = \beta\,|0\rangle + \alpha\,|1\rangle\,. \quad (3)$$

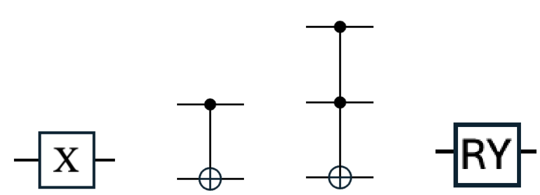
The control-NOT (cX) gate is a two-dimensional quantum transformation with one control qubit and one target qubit. When the control qubit is $|1\rangle$, the target qubit is flipped by the NOT gate; otherwise, the target qubit remains unchanged.

The Toffoli gate (T) is a three-dimensional quantum transformation with two control qubits and one target qubit. When both control qubits are $|1\rangle$, the target qubit is flipped by the NOT gate; otherwise, the target qubit remains unchanged.

The y-axis rotation gate (RY) is used to mix the amplitudes of $|0\rangle$ and $|1\rangle$ without introducing a relative phase. In the quantum circuit, the matrix representation of the RY gate is

$$RY(\theta) = \begin{bmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{bmatrix}. \quad (4)$$

The circuit representations of the Not gate, the control-NOT gate, the Toffoli gate and the RY gate are shown in Fig. 1.



(a) X (b) cX (c) T (d) RY

Fig. 1: Circuit representations of frequently used Quantum gates.

2) Measurement: The only way to extract information from qubits is by making a measurement [17]. For a single qubit, the outcome of one measurement is either 0 or 1. For an n-qubit, the outcome of one measurement is an n-bit string. For example, a bit string 00010 is a measurement of a 5-qubit. More generally, if the state of an n-qubit is

$$|\Psi\rangle_n = \sum_{0 \leq x < 2^n} \alpha_x |x\rangle_n , \tag{5}$$

the probability that the zeros and ones resulting from measurements of all qubits are

$$p_x(|\Psi\rangle_n) = |\alpha_x|^2. \tag{6}$$

This basic rule was first enunciated by Max Born and is known as the *Born rule* [17].

Let U be a quantum circuit composed of n-qubit, the measurement of U is an n-bit. Here, Qiskit [10] provides a way to obtain the probability $p_x(U)$. The general idea of Qiskit is to simulate multiple measurements and the probability $p_x(U)$ is calculated as the count of x. For example, let U_2 be a quantum circuit composed of 2-qubit, we want to know the probability $p_{01}(U_2)$. We simulate the measurement of U_2 5 times and the outcomes are $\{01, 00, 10, 01, 11\}$, then $p_{01}(U_2)$ is calculated as $\frac{2}{5} = 0.4$.

In addition, Qiskit can obtain the probability of the measurement outcome 1 from a single qubit within a quantum circuit as $p_1(|\psi\rangle)$, where ψ is a qubit. The general way to obtain this probability is to calculate the count of 1 of the i^{th} bit in the bit string if ψ is the i^{th} qubit in the quantum circuit. Following the same example, let ψ be the second qubit, then $p_1(\psi) = \frac{3}{5} = 0.6$.

III. Quantum Fuzzy Measure

A. *Quantum Fuzzy Measure Definition*

Let $|g\rangle$ be a set function that maps the power set of S to the given qubits as

$$|g\rangle(A) = |\psi_A\rangle = \alpha_A |0\rangle + \beta_A |1\rangle , \tag{7}$$

where, $A \subseteq S$, $|\alpha_A|^2 + |\beta_A|^2 = 1$. We refer to $|g\rangle$ as the quantum fuzzy measure (QFM). The QFM follows these constraints:

1) Boundary conditions:
   $|g\rangle(\varnothing) = \mathbf{1}|0\rangle + \mathbf{0}|1\rangle = |0\rangle$,
   $|g\rangle(S) = \mathbf{0}|0\rangle + \mathbf{1}|1\rangle = |1\rangle$
2) Monotonicity:
   If $A \subseteq B \subseteq S$, then $|\beta_A|^2 \leq |\beta_B|^2$

B. *QFM Parametrization*

Let $|g\rangle$ be a QFM and $A \subseteq S$, according to (4) and (7), the RY gate is employed to encode $|g\rangle$ as

$$|g\rangle(A) = RY(\theta_A)|0\rangle = cos(\frac{\theta_A}{2})|0\rangle + sin(\frac{\theta_A}{2})|1\rangle . \tag{8}$$

Let $A \subseteq B \subseteq S$, to follow the monotonicity constraint, θ_A and θ_B are set as

$$\theta_A,\ \theta_B \in [0, \pi]\ and\ \theta_A \leq \theta_B. \tag{9}$$

A QFM can be parametrized by θ refers to (8) and (9), however, if we measure this QFM, the monotonicity may break in the sense of BFM. We use an example to present this phenomenon.

Set $\theta_A = \frac{\pi}{3}$ and $\theta_B = \frac{2\pi}{3}$, then $|g\rangle(A) = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$ and $|g\rangle(B) = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$. Refer to the monotonicity constraint in Section III-A, $|\beta_A|^2 = \frac{1}{4}$, $|\beta_B|^2 = \frac{3}{4}$, then $|\beta_A|^2 \leq |\beta_B|^2$, so $|g\rangle(A)$ and $|g\rangle(B)$ follow the monotonicity constraint of QFM.

However, as stated in [17]: '*Making a measurement consists of performing a certain test on each Qubit, the outcome of which is either 0 or 1.*', if we make a measurement on these two qubits, the outcome could be $\{1, 0\}$, where the measurement of $|g\rangle(A)$ is 1 and the measurement of $|g\rangle(B)$ is 0. In this case, let g be the QFM corresponding to this measurement of $|g\rangle$; g does not follow the monotonicity constraint of the numeric FM as introduced in Section II-A, as $g(A) = 1$, $g(B) = 0$, $g(A) > g(B),\ A \subseteq B$.

Here, a quantum gate is proposed to ensure that if we make a measurement on a QFM, the corresponding QFM also follows the monotonicity constraint. The monotonicity constraint gate (MC) is a five-dimensional quantum transformation with four control qubits and one target qubit. When the first or the second qubit is $|1\rangle$, the target qubit is set as $|1\rangle$; otherwise, the target qubit remains unchanged. Let $A \subset C \subseteq S$, where $|g\rangle(A)$, $|g\rangle(B)$, $|g\rangle(C)$ are three QFMs and $B = C \setminus A$. Fig. 2 shows the structure of MC and Table I shows the evolution of the superposition in one measurement, where 'Input' and 'Output' represent the combinations of superposition of $|g\rangle(A)$, $|g\rangle(B)$ and $|g\rangle(C)$, respectively.

Fig. 3 shows the QFM lattice of the Input 011, where the worth at the white node is in the state $|0\rangle$ and the worths at the black nodes are in the state $|1\rangle$. It is shown that the measurement outcomes, i.e. the QFMs, of all possible outputs follow the monotonicity constraint of the numeric FM. The circuit representation of this gate is shown in Fig. 4, since we do not need to measure the third and the fifth qubit.

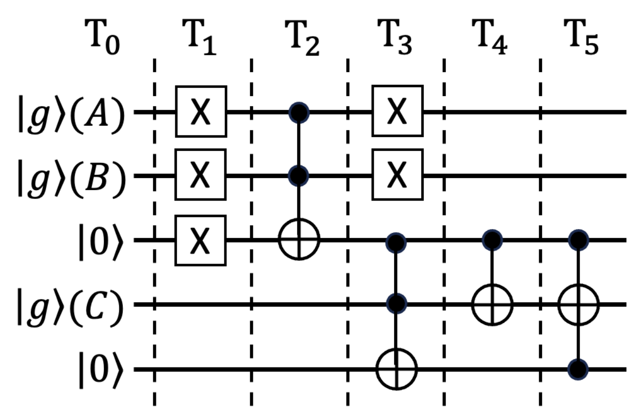


Fig. 2: The monotonicity constraint gate (MC).

The MC gates are used to ensure that the measured worths in the QFM satisfy monotonicity. To enforce monotonicity over the entire QFM lattice, we propose a quantum circuit

TABLE I: Evolution of superposition quantum registers for monotonicity constraint gate (MC)

Input A, B, C	T_0	T_1	T_2	T_3	T_4	T_5	Output A, B, C
000	00000	11100	11000	00000	00000	00000	000
001	00010	11110	11010	00010	00010	00010	001
010	01000	10100	10100	01100	01110	01110	011
011	01010	10110	10110	01111	01101	01111	011
100	10000	01100	01100	10100	10110	10110	101
101	10010	01110	01110	10111	10101	10111	101
110	11000	00100	00100	11100	11110	11110	111
111	11010	00110	00110	11111	11101	11111	111

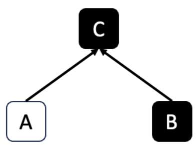


Fig. 3: The QFM lattice for the Input 011 in Table I.

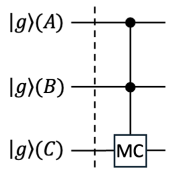


Fig. 4: Circuit representation of the MC gate.

composed of RY gates and MC gates, which also enables parametrization of the QFM. We refer to this circuit as $Q(\theta^*)$, where $\theta^* = \{\theta^*_A \mid \theta^*_A \in [0, \pi],\ A \subseteq S\}$ to district from θ in (8) and (9).

Here, we can transform θ^* to θ as

$$sin^2(\frac{\theta_{\{s_i\}}}{2}) = \\ 1 - \prod_{x_i \in A_k} \left(1 - sin^2(\frac{\theta^*_{\{s_i\}}}{2})\right) \prod_{n=2}^{k} \left(1 - sin^2(\frac{\theta^*_{\bigcup_{j=1}^{n}\{s_{(j)}\}}}{2})\right), \tag{10}$$

where $A_k = \{s_{(1)}, \cdots, s_{(k)}\}$, $k \leq N$ is one of the combinations of k individual sources and $s_{(j)}$ represents the j^{th} sources in A_k. Then we can obtain the parametrized QFM according to (8). For example, $\theta^* = \{\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\}$ can be transformed as $\theta = \{\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, 2\arcsin(\sqrt{\frac{7}{8}}), 2\arcsin(\sqrt{\frac{7}{8}}), 2\arcsin(\sqrt{\frac{7}{8}})\}$. And the QFM encoded by θ follows the monotonicity constraint of the QFM referring to (8).

Fig. 5 shows the quantum circuit representation for a three-source QFM as an example. Note that we do not apply the RY gate to the last qubit in the quantum circuit, since it is $|g\rangle\,(S)$ and is always set as $|1\rangle$.

C. QFM Measurement

As $Q(\theta^*)$ enables parametrization of the QFM, the measurement of a QFM (i.e. a BFM) can be represented by the measurement of the $Q(\theta^*)$. So, to obtain a measurement for a QFM, we can measure $Q(\theta^*)$ and then transform it into a BFM.

Let x be a possible measurement outcome of $Q(\theta^*)$, which is a $(2^I - 1)$-bit string. This $(2^I - 1)$-bit string is a representation of a measurement outcome of a QFM for I sources. For example, let x be 1001101, which is a 7-bit string. It is

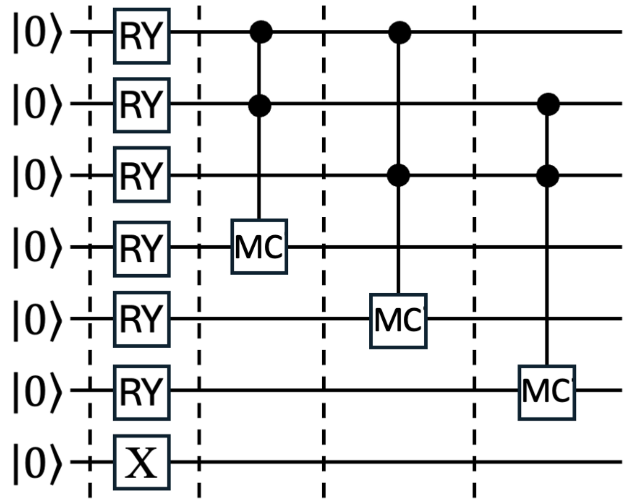


Fig. 5: $Q(\theta^*)$ Quantum circuit for three-source QFM.

a representation of a possible outcome of a QFM for $I = 3$ sources, which is a BFM as shown in Table. II. As discussed in Section II-C2, the probability of obtaining this outcome in one measurement is $p_x(Q(\theta^*))$ and this paper uses Qiskit to obtain it.

Following the example as Fig. 5, let $\theta^* = \{\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}\}$, θ and the corresponding QFM can be obtained according to (8) and (10). Fig. 6 shows the counts of all possible measurement outcomes in 20000 measurements. The probability of a measurement outcome of the corresponding QFM can be obtained by the corresponding count divided the total numer of measurements (20000 in this case).

For example, the probability of obtaining the measurement outcome 1001101 is $p_{1001101}(Q(\theta^*)) = \frac{2455}{20000} = 0.12275$. Here, the measurement outcome 1001101 is a representation of a BFM $g_{1001101}(\theta^*)$ as shown in Table. II, where this BFM is one possible measurement of the QFM parametrized by $Q(\theta^*)$.

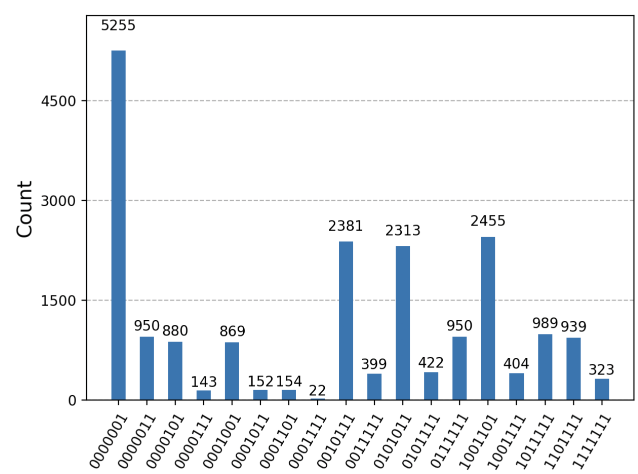


Fig. 6: An example of all possible measurements for a three-source QFM.

TABLE II: The BFM $g_{1001101}(\theta^*)$ corresponds to the measurement outcome 1001101

$s_1 : 1$	$s_1, s_2 : 1$	$S : 1$
$s_2 : 0$	$s_1, s_3 : 1$	
$s_3 : 0$	$s_2, s_3 : 0$	

Furthermore, we can obtain the probability of obtaining the measurement outcome 1 from a single qubit within $Q(\theta^*)$ as $p_1(|g\rangle(A))$ by Qiskit as discussed in Section II-C2. Following the same example,

$$p_1(|g\rangle(\{s_1, s_2\})) = \frac{10391}{20000} = 0.51955.$$

D. QFM Interpretability

A BFM captures the information of one 0/1 configuration where the worths of it are either 0 or 1. If it is not certain whether the worths are 0 or 1, qubits are applied to describe the worths, turning the BFM into a QFM. In reference to the interpretation of a BFM in [9], each worth in the QFM is either 0 or 1, indicating that the corresponding combination of sources could be insufficient or worth to be considered. It is unknown whether it is sufficient or insufficient, unless we make a measurement on it. Looking at the entire QFM lattice, in reference to Fig. 6, we can see that a QFM contains information about all possible BFMs, each associated with a probability, in the sense of a single measurement.

Beyond describing the worths for one configuration, the QFM systematically explores the full configuration space and assigns probabilities to configurations, where those with higher probabilities indicate that they are more sufficient to describe the worths in the corresponding BFMs. In other words, it helps us to choose which BFM could be the best to capture the worths of combinations of sub-sources. For example, as shown in Fig. 6, the measurement outcome 0000001 has the highest probability as 26.275%. Assume this is a three-sensor fusion task, it indicates that it is the most sufficient to combine all three sensors to finish a task.

IV. Fusing Information by the CFI in Respect to the QFM

In the previous section, we introduced the QFM which uses qubits to capture the worths of combinations of sub-sources. Here, it is natural to fuse information using the CFI based on the QFM and to explore the results of the CFI. Referring to (1), (5) and (6)

$$C_{|g\rangle}(h) = \sum_{0 \le x < 2^n} p_x(|\Psi\rangle_n) C_{g_x}(h), \tag{11}$$

where g_x represents the BFM corresponding to bit-string x, $|\Psi\rangle$ represents $Q(\theta^*)$ and $n = 2^I - 1$. To align with (5) and (6), we use $|\Psi\rangle$ instead of $Q(\theta^*)$. For example, if $x = 1001101$, then g_x is the BFM in Table. II. As discussed in Section II-C2, $p_x(|\Psi\rangle_n)$ can be obtained by Qiskit.

Although $C_{|g\rangle}(h)$ is not an n-qubit, it can be interpreted by referring to the measurements of an n-qubit. The term $C_{g_x}(h)$ can be interpreted as the CFI based on a possible measurement of the QFM $|\Psi\rangle$, which can be represented by a bit-string x. The term $p_x(|\Psi\rangle_n)$ can be interpreted as the probability of obtaining a measurement of $|\Psi\rangle$ as the BFM represented by x. The overall equation can be interpreted as the probability-weighted sum, i.e. the expectation, of the CFI based on all possible measurements of $|\Psi\rangle$.

Following the example in Fig. 6, let $h = \{3, 2, 1\}$ be a three-source evidence, Fig. 7 shows the counts of $C_{|g\rangle}(h)$, respectively. Here, the CFI can be interpreted as

$$C_{|g\rangle}(h) = 0.48045 \cdot \mathbf{1} + 0.26405 \cdot \mathbf{2} + 0.2555 \cdot \mathbf{3},$$

where the bold numbers represent the possible values of $C_{|g\rangle}(h)$, which are the CFI based on all possible measurements of $|\Psi\rangle$, and the probabilities of $C_{|g\rangle}(h)$ equal to $\{1, 2, 3\}$ are $\{48.045\%,\ 26.405\%,\ 25.55\%\}$, respectively. The sum of all terms can be interpreted as the expectation of $p_x(|\Psi\rangle_n) C_{g_x}(h)$ as 1.77505.

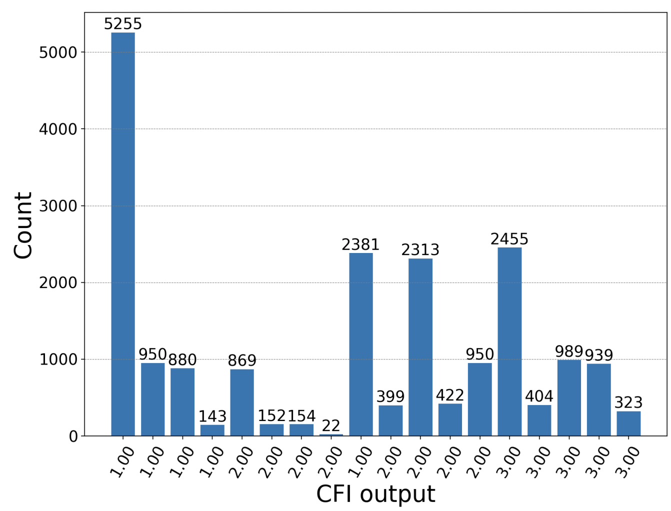


Fig. 7: The counts of $C_{|g\rangle}(h)$ in respect to Fig. 6.

The above example only contain three possible values of the CFI outputs, here, a synthetic scenario is formed following the example above as a three-sensor fusion task to demonstrate the situation with more possible values, where $h_{s_i,t}$ denotes the output of three distance sensors and t indicates that the data are collected over time.

The dataset is generated by grid sampling of three sensor outputs, each uniformly sampled from $[0, 10]$ with 10 equally spaced values. This results in $10 \cdot 10 \cdot 10 = 1000$ samples. Fig. 8 shows the counts of the average of the CFI outputs across the dataset as

$$\bar{C}_{|g\rangle}(h_S) = \frac{\sum_{t=1}^{1000} C_{|g\rangle}(h_{S,t})}{1000},$$

where $C_{|g\rangle}(h_{S,t})$ represents the CFI output at time t. It can be interpreted as

$$\bar{C}_{|g\rangle}(h_S) = 0.26275 \cdot \mathbf{2.25} + 0.13495 \cdot \mathbf{3.17} + 0.02245 \cdot \mathbf{4.08} + 0.35855 \cdot \mathbf{5.00} + 0.06125 \cdot \mathbf{5.92} + 0.1439 \cdot \mathbf{6.83} + 0.01615 \cdot \mathbf{7.75},$$

where $\bar{C}_{|g\rangle}(h_{S,t})$ is rounded to two decimal places, the bold numbers represent the possible values of $\bar{C}_{|g\rangle}(h_S)$, and the probability that it is equal to 2.25 is 26.275% and the others are similar. The sum of all terms can be interpreted as the expectation of the average of the CFI based on one measurement outcome across the dataset as 4.37.

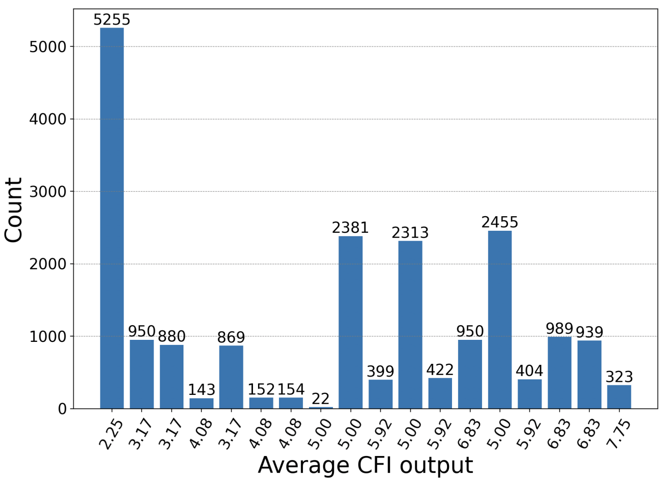


Fig. 8: The counts of the average of the CFI outputs across the synthetic dataset in respect to Fig. 6.

V. Conclusions & Future work

This paper sets out to explore a new interpretation of the FM based on quantum computing where we use qubits to represent the worths of the FM. We define the boundary conditions and the monotonicity constraint of the QFM. A quantum circuit ($Q(\theta^*)$), composed of RY gates and monotonicity constraint gates (MC), is used to ensure that the measurement of a QFM, i.e. a BFM, follows the monotonicity of the numeric FM. We analyse the measurement of the QFM and use the measurement of $Q(\theta^*)$ to represent it. We articulate that measuring a QFM yields all possible 0/1 configurations in the sense of a BFM, each assigned a probability estimated by Qiskit. Finally, we discuss the interpretation provided by the QFM compared to the BFM, showing that the QFM can be interpreted as containing the information of all possible BFMs.

Furthermore, this paper explores the CFI based on the QFM in the context of information aggregation. We show that the CFI regarding the QFM can be interpreted as the CFI based on all possible measurements of the QFM, in reference to the quantum mechanism. The CFI based on the QFM also acts as the probability-weighted sum, i.e. expectation, of the CFI based on all possible measurements of the QFM. A synthetic example is used alongside the introduction of the proposed approach.

One of the limitations of this paper is that the MC gate changes some QFMs by forcing the measurement of them to follow the monotonicity constraint of the numeric FM, even though they follow the monotonicity constraint of the QFM. It is worth exploring these QFMs in future work. In addition, we only provide a synthetic example in this paper. It could be interesting to explore the QFM in the context of ensemble classifiers in real-world situations in the future.

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