Measure derivatives induced by the Maximal chain-based integral

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***Abstract*—Fuzzy measures and integrals are used in a wide range of applications related to decision making and information fusion. A key concept of measure theory is the notion of derivatives. For example, the Radon-Nikodym derivative between two additive measures is the function that, when integrated with respect to one measure, produces the second one. In this paper, we consider the generalized concept of derivatives for set-valued integrals. More particularly, we study the derivatives induced by a Maximal chain-based integral.**

***Index Terms*—Fuzzy measures, Measure derivatives, Fuzzy integrals, Choquet integral, Maximal chain-based integral.**

I. Introduction

Fuzzy measures and integrals provide a flexible aggregation framework that models interactions among information sources or criteria, enabling representations that cannot be captured by classical additive concepts. As a result, they are used in multiple practical problems, including decision making and information fusion.

Mathematically speaking, a fuzzy measure is a set function that is monotone with respect to set inclusion. In general, an integral that allows to integrate a function with respect to a fuzzy measure, is called a fuzzy integral [17], [18]. The two most well-known fuzzy integrals are the Choquet [2] and the Sugeno [15] integral. Moreover, there exist several generalizations as well [5], [12], [13].

An important concept associated with integrals is the notion of a measure derivative. Given two measures ν and μ, the derivative of ν with respect to μ is the function that, when integrated with respect to μ, produces ν. When μ and ν are additive and the Lebesgue integral is considered, the function is called the Radon-Nikodym derivative. This concept of derivatives has been extended to fuzzy measures by considering fuzzy integrals. Several authors [3], [8], [10], [11], [14], [16] have contributed to this topic, mostly assuming the Choquet integral. Derivatives of additive measures are a key concept in measure theory, as they are basic components for building distances (as e.g., f-divergence, KL-divergence, and Hellinger distance) as well as for defining entropy for measures. In this regard, we expect the derivatives of fuzzy measures to have the same relevance.

Islam et al. [6] have shown that the Choquet integral can be represented as a multi-layer network. In this case, the network has multiple inputs, a hidden layer, and a single output. In our work, we extend this representation by considering aggregation methods based on the Choquet integral that allows multiple outputs in the output layer.

Our integral construction is related to set-valued functions, inspired by Aumann [1]. Such integrals can be seen as a natural generalization of both the integral of point-valued functions and of the sum of a finite number of sets. The motivation for this approach lies in its connection with statistical and economic problems. The integral is then used to introduce the corresponding measure derivatives. This generalised insight into the concept of derivatives can offer a fresh perspective and yield results where traditional direct methods have limitations.

The structure of the paper is as follows. In Section II, the Maximal chain-based integral is proposed. It is related to [5], which introduces a more general Maximal chain-based Choquet-like integral. We link our definition with the representation of an integral proposed by Islam et al. [6] as a multi-layer network. Then, a generalization is provided that produces a set-valued integral, together with the link to a multi-layer network with multiple outputs. Section III studies the derivative of the previously introduced integral, and includes some examples. The paper finishes with concluding remarks and directions for future research.

II. The integral

Let X be a reference set and, since working with sets, $\subset$ our binary relation on X. Then X is an ordered set because for all $X_1, X_2, X_3 \in X$, the $\subset$ relation satisfies properties of reflexivity ($X_1 \subset X_1$), antisymmetry ($[X_1 \subset X_2 \wedge X_2 \subset$

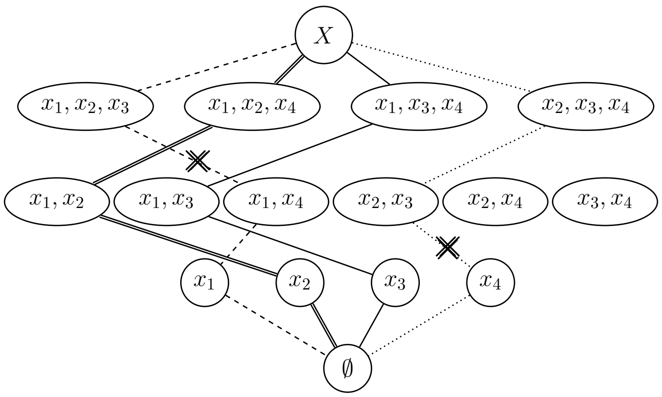

Fig. 1. Chains and non-chains for reference set $X = \{x_1, x_2, x_3, x_4\}$

$X_1] \Rightarrow X_1 = X_2$) and transitivity ($[X_1 \subset X_2 \wedge X_2 \subset X_3] \Rightarrow X_1 \subset X_3$). An ordered set X is called a chain if and only if all its elements are comparable with respect to $\subset$ relation, i.e., for all $X_1, X_2 \in X$ it holds either $X_1 \subset X_2$ or $X_2 \subset X_1$.

Definition 1. *Let X be a finite reference set, then $\mathcal{C}$ is a maximal chain if $\mathcal{C} := \{C_1, C_2, \dots, C_m\}$ is a chain in X such that $\emptyset = C_0 \subset C_1 \subset \cdots \subset C_m = X$ and $C_k \setminus C_{k-1} = \{x_k\}$ for all $k = 2, \dots, m$.*

We denote by $\mathcal{M}(X)$ the set of all maximal chains of X. The following example illustrates the concept of maximal chains.

Example 1. *Let $X = \{x_1, x_2, x_3, x_4\}$. Then the corresponding maximal chains can be*

$$C_1 = \{\emptyset, \{x_3\}, \{x_1, x_3\}, \{x_1, x_3, x_4\}, X\}$$
$$C_2 = \{\emptyset, \{x_2\}, \{x_1, x_2\}, \{x_1, x_2, x_4\}, X\}$$

but not

$$C_3 = \{\emptyset, \{x_1\}, \{x_1, x_4\}, \{x_1, x_2, x_3\}, X\}$$
$$C_4 = \{\emptyset, \{x_4\}, \{x_2, x_3\}, \{x_2, x_3, x_4\}, X\},$$

because $\{x_1, x_4\} \not\subset \{x_1, x_2, x_3\}$ in C_3 and $\{x_4\} \not\subset \{x_2, x_3\}$ in C_4, as illustrated in Figure 1.

Definition 2. *[15] A fuzzy measure μ is a set function $\mu : 2^X \to [0, \infty)$ satisfying $\mu(\emptyset) = 0$ and $\mu(A) \leq \mu(B)$ for $A \subset B$.*

Definition 3. *[2] Let $f : X \to \mathbb{R}^+$ with $f(x_1) \geq f(x_2) \geq \cdots \geq f(x_m)$, μ be a fuzzy measure and sets $C_0 = \emptyset, C_k = \{x_1, x_2, \dots, x_k\}$ for $k = 1, 2, \dots, m$. Then the Choquet integral of f with respect to μ is defined by*

$$(C) \int f d\mu = \sum_{k=1}^{m} f(x_k)[\mu(C_k) - \mu(C_{k-1})].$$

Definition 4. *Let X be a finite reference set, $\mathcal{M}(X)$ the set of all the possible maximal chains on X, $f : X \to \mathbb{R}^+$ a function, and μ a fuzzy measure. Then, for any $\mathcal{C} \in \mathcal{M}(X)$, we define the Maximal chain-based integral as*

$$(\mathrm{MC}) \int f d\mu = \sum_{k=1}^{m} f(x_k)[\mu(C_k) - \mu(C_{k-1})].$$

The set of all the possible Maximal chain-based integrals for a given f and μ, denoted as $\mathcal{S}(f, \mu)$, is defined as

$$\mathcal{S}(f, \mu) = \{(\mathrm{MC}) \int f d\mu \mid \mathcal{C} : \textit{maximal chain on } X\}.$$

The notation and naming of the integral above are adopted from [5], where a more general Maximal chain-based Choquet-like integral is introduced. It also uses maximal chains as the integration sets, but a binary operation $\otimes$ is utilized as an aggregation operator between the function and the measure, while we use only multiplication.

Figure 2 illustrates the computation of $\mathcal{S}(f, \mu)$. There, the input layer consists of singletons in X. The hidden layer (node) h_j represents applying the function and the measure to all possible maximal chains of the given reference set X, so $h_j = \mathcal{S}(f, \mu)$. The output layer, with nodes denoted as y_i, corresponds to the final results after the integration, one for each maximal chain. Hence, i as the number of outputs satisfies $i = |\mathcal{S}(f, \mu)|$.

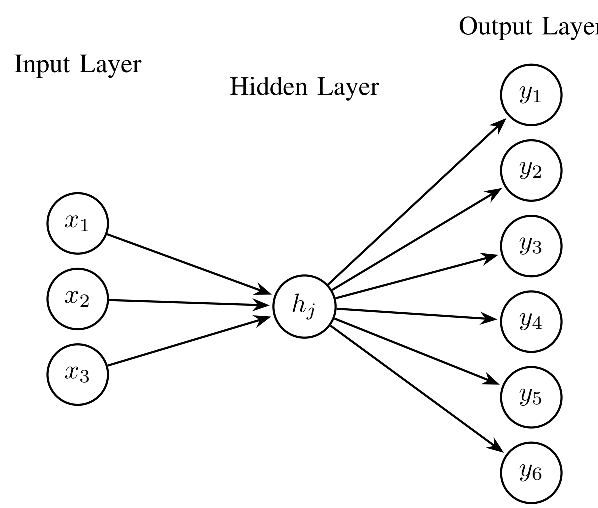

Fig. 2. Graphical representation of the computation of $\mathcal{S}(f, \mu)$.

It is easy to see that the presented integral is additive for a given $\mathcal{C}$. The reason is measuring the same set of the chain for the corresponding singleton regardless of the function.

Proposition 1. *For a given $\mathcal{C}$,*

$$(\mathrm{MC}) \int (f + g) d\mu = (\mathrm{MC}) \int f d\mu + (\mathrm{MC}) \int g d\mu.$$

A. Particular cases

By construction of the Choquet integral, it can be considered a special case of the introduced concept.

Proposition 2. *For any f and μ, the corresponding Choquet integral satisfies*

$$(C)\int f d\mu \in \mathcal{S}(f,\mu).$$

For special classes of measures and some particular measures, the original concept reduces to the less complex ones. In the following, we look at the cardinality of the corresponding $\mathcal{S}(f,\mu)$. Recall that the dual measure μ^C to a measure μ is defined for all $A \in X$ as $\mu^C(A) = \mu(X) - \mu(X \setminus A)$.

Proposition 3. *Taking an arbitrary measure μ and μ^C as its dual, then*

$$|\mathcal{S}(f,\mu)| = \left|\mathcal{S}\left(f,\mu^C\right)\right|$$

Proposition 4. *If μ is additive, then*

$$|\mathcal{S}(f,\mu)| = 1.$$

Figure 3 represents the computation for all the cases with $|\mathcal{S}(f,\mu)| = 1$, where we only get one resulting value y.

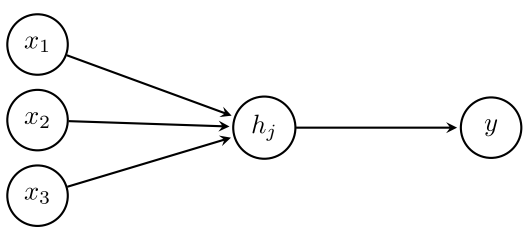

Fig. 3. Graphical representation of the $\mathcal{S}(f,\mu)$ computation with restrictions.

Proposition 5. *Let us assume Dirac measure in the form* $\delta_{x_0}(A) = \begin{cases} 1 & x_0 \in A \\ 0 & x_0 \notin A \end{cases}$ *and an arbitrary function f. Then*

$$|\mathcal{S}(f,\delta_{x_0})| = 1.$$

Moreover, $\exists \alpha \in \{1,\ldots,m\}$ such that $C_\alpha \setminus C_{\alpha-1} = \{x_0\}$ and thus $\mathcal{S}(f,\delta_{x_0}) = \{f(x_0)\}$ for all the maximal chains.

Proof.

$$\begin{aligned}\mathcal{S}(f,\delta_{x_0}) &= \left\{\sum_{k=1}^{m} f(x_k)[\mu(C_k) - \mu(C_{k-1})]\right\} \\ &= \left\{\sum_{k=1}^{\alpha-1} f(x_k)(0-0) + f(x_\alpha)(1-0) + \right. \\ &\qquad \left. + \sum_{k=\alpha+1}^{m} f(x_k)(1-1)\right\} = \{f(x_0)\}.\end{aligned}$$

Since the same holds for every maximal chain, then $|\mathcal{S}(f,\delta_{x_0})| = 1$. □

B. Symmetric fuzzy measures

A fuzzy measure is said to be symmetric on 2^X if $|A| = |B|$ implies $\mu(A) = \mu(B)$ for all $A, B \in X$. The next proposition follows from the rearrangement inequality [4].

Proposition 6. *Let μ be a symmetric fuzzy measure, $\mathcal{C} := \{C_k | k = 0,1,\ldots,m\} \in \mathcal{M}(X)$ and $\Delta_k := \mu(C_k) - \mu(C_{k-1})$.*

1) *If Δ_k is monotone decreasing, then*

$$(C)\int f d\mu = \max_{C \in \mathcal{M}(X)} \mathcal{S}(f,\mu).$$

2) *If Δ_k is monotone increasing, then*

$$(C)\int f d\mu = \min_{C \in \mathcal{M}(X)} \mathcal{S}(f,\mu).$$

Example 2. *Let $X := \{x_1, x_2, x_3\}$ and symmetric fuzzy measure μ is defined as follows:*
$\mu(\{x_1\}) = \mu(\{x_2\}) = \mu(\{x_3\}) = 1$
$\mu(\{x_1,x_2\}) = \mu(\{x_2,x_3\}) = \mu(\{x_1,x_3\}) = 3$
$\mu(\{x_1,x_2,x_3\}) = 6.$
Then $\Delta_1 = 1, \Delta_2 = 2, \Delta_3 = 3$, so Δ_k is monotone increasing.
Now, we define f as $f(x_1) = 1, f(x_2) = 4, f(x_3) = 9$.
For chain $C_3 : \emptyset \subset \{x_2\} \subset \{x_1,x_2\} \subset \{x_1,x_2,x_3\}$ it holds

$$\begin{aligned}(\mathrm{MC}_3)\int f d\mu &= f(x_2)[\mu(\{x_2\}) - \mu(\emptyset)] \\ &+ f(x_1)[\mu(\{x_1,x_2\}) - \mu(\{x_2\})] \\ &+ f(x_3)[\mu(X) - \mu(\{x_1,x_2\})] = 33.\end{aligned}$$

In the same way, we have the next table

Chain				MC-*integral*
C_1 :	x_1	x_1,x_2	x_1,x_2,x_3	36
C_2 :	x_1	x_1,x_3	x_1,x_2,x_3	31
C_3 :	x_2	x_1,x_2	x_1,x_2,x_3	33
C_4 :	x_2	x_2,x_3	x_1,x_2,x_3	25
C_5 :	x_3	x_1,x_3	x_1,x_2,x_3	23
C_6 :	x_3	x_2,x_3	x_1,x_2,x_3	20

Since $f(x_1) < f(x_2) < f(x_3)$, the Choquet integral corresponds to the (MC)*-integral with C_6, which is also the chain corresponding to the minimal resulting value after integration as claimed in Proposition 6. That is,*

$$(C)\int f d\mu = (\mathrm{MC}_6)\int f d\mu = \min_{\mathcal{C}} \mathcal{S}(f,\mu).$$

C. 2-additive fuzzy measures

Definition 5. *The Möbius transform of a fuzzy measure μ on X is a function $m : 2^X \to R$ such that for any $A \in 2^X$*

$$m(A) := \sum_{K \subset A} (-1)^{|A \setminus K|} \mu(K).$$

Definition 6. *[7] A fuzzy measure μ on X is said to be 2-additive if for all $A \subset X, |A| > 2$ it holds $m(A) = 0$, and there exists $B \subset X$ such that $|B| = 2$ and $m(B) \neq 0$.*

Proposition 7. *[7] Let μ be a 2-additive fuzzy measure on X. Then for all $K \subset X$ such that $|K| \geq 2$,*

$$\mu(K) = \sum_{\{x_i,x_j\}\subset K} \mu(\{x_i, x_j\}) - (|K| - 2) \sum_{x_i \in K} \mu(\{x_i\}).$$

Using this proposition, the next result for (MC)-integral can be proposed.

Proposition 8. *Let μ be a 2-additive fuzzy measure on 2^X and $f : X \to R^+$. For $k \in \mathbb{N}, 1 \leq k \leq m$, consider the chain $\{x_1\} \subset \{x_1, x_2\} \subset \cdots \subset \{x_1, x_2 \dots x_k\} \subset \cdots \subset X$ and define*

$$\Delta_k = \sum_{i \neq k} (\mu(\{x_i, x_k\}) - \mu(\{x_i\})) - (k - 2)\mu(\{x_k\}).$$

Then we have

$$(\mathrm{MC}) \int f d\mu = \sum_k f(x_k)\Delta_k.$$

Even for 2-additive fuzzy measure, $|\mathcal{S}(f, \mu)| \leq m!$ in general.

D. Generalizations

Definition 7. *Let X be a reference set, $\mathcal{M}(X)$ a set of all maximal chains in X, μ be a fuzzy measure and $f : X \to R^+$ an arbitrary function. Then the averaging functional of all the corresponding (MC)-integrals is defined as*

$$A(f, \mu) := \frac{1}{|\mathcal{M}(X)|} \sum_{\mathcal{C}\in\mathcal{M}(X)} (\mathrm{MC}) \int f d\mu.$$

This generalization allows to look at the comparison between (MC)-integral and already known ones, taking the particular type of measure. The key difference is that mostly integrals use only the information from one chain, while $A(f, \mu)$ can be considered the centroid or the average of all chains, using all information.

The next proposition follows from Proposition 4 on additive measures. Recall that weaker version of additivity are subadditivity with the inequality $\mu(A \cup B) \leq \mu(A) + \mu(B)$ for all $A, B \in X$, and superadditivity with the same inequality only the opposite inequality sign.

Proposition 9. *If μ is additive,*

$$A(f, \mu) = \int f d\mu.$$

After weakening additivity condition, if μ is subadditive

$$A(f, \mu) \leq \int f d\nu$$

and if μ is superadditive

$$A(f, \mu) \geq \int f d\nu.$$

The integral on the right is the classic Lebesgue integral. Integration measure ν is an additive measure built on singletons as $\nu(\{x_i\}) = \mu(\{x_i\})$ for all i.

For additive measures, the result is easy to see since $\mu(C_k) - \mu(C_{k-1}) = \mu(C_k \setminus C_{k-1}) = \mu(\{x_k\})$. In case of sub- or superadditive measure, only the first equality changes to the corresponding inequality. Thus, (MC)-integral is comparable with the additive Lebesgue integral.

Definition 8. *Let X be a reference set, μ a fuzzy measure, and let $F(X) := \{f_1, f_2 \dots, f_l\}$ be a finite family of non-negative functions. Then, we define family-induced set of all possible Maximal chain-based integrals, denoted as $\mathcal{S}(F, \mu)$, as follows*

$$\mathcal{S}(F, \mu) := \bigcup_{f\in F} \mathcal{S}(f, \mu).$$

Figure 4 represents the computation of $\mathcal{S}(F, \mu)$.

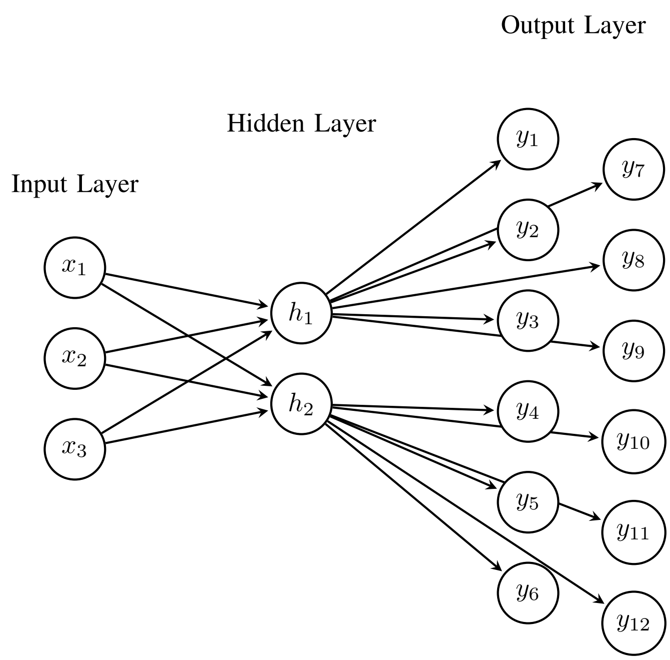

Fig. 4. Graphical representation of the computation of $\mathcal{S}(F, \mu)$.

III. The derivative

In this section, we look at the corresponding derivative for the Maximal chain-based integral. The problem is inverse to the original integral computation. Here, the integration measure μ is given together with the resulting measure ν, and the task is finding the integration function f. Symbolically written

$$\nu(A) = (\mathrm{MC}) \int_A f d\mu \qquad \Leftrightarrow \qquad f = \frac{d\nu}{d\mu} \text{ on } A.$$

So far, the (MC)-integral is only defined for the whole reference set X. For the purpose of the derivatives, we need to consider it on any subset $A \subset X$.

Definition 9. *Let μ be a fuzzy measure, $\mathcal{C}$ a maximal chain on X, such that $\mathcal{C} : \emptyset = C_0 \subset C_1 \subset \cdots \subset C_m = X$. Let $A \in \mathcal{C}$. Then there exists $m_0 \leq m$ such that $A = C_{m_0}$,*

$\emptyset = C_0 \subset C_1 \subset \cdots \subset C_{m_0} \subset \cdots \subset C_m = X$ *and the* (MC)-*integral of f with respect to μ restricted to A is defined by*

$$\text{(MC)} \int_A f d\mu = \sum_{k=1}^{m_0} f(x_k)[\mu(C_k) - \mu(C_{k-1})].$$

Proposition 10. *Let X, $\mathcal{C}$, f and μ be defined as above. Then,*

$$\text{(MC)} \int_A f d\mu = \text{(MC)} \int f_A d\mu$$

for all $A \in \mathcal{C}$, with $f_A(x) = \begin{cases} f(x) & x \in A \\ 0 & x \notin A \end{cases}$ (also $f_A(x) = f(x) \cdot \mathbf{1}_A$).

Now, we can closely look at the derivative itself, particularly the existence condition and the definition adapted for the chains.

Let $C_1 = \{x_1\}, C_2 = \{x_1, x_2\}, \ldots, C_m = X$. Also suppose fuzzy measure μ such that $0 < \mu(C_1) < \mu(C_2) < \cdots < \mu(C_m) = \mu(X)$ and another fuzzy measure ν.
Then we define the function $f : X \to R^+$ as follows:

$f(x_1) = \nu(C_1)/\mu(C_1)$,
$f(x_2) = (\nu(C_2) - \nu(C_1))/(\mu(C_2) - \mu(C_1))$
...
$f(x_k) = (\nu(C_k) - \nu(C_{k-1}))/(\mu(C_k) - \mu(C_{k-1}))$ for $k = 1, 2, \ldots, m$.

Then, for any $C_{m_0} \in \mathcal{C}$, it can be written $\nu(C_{m_0}) = \sum_{k=1}^{m_0} f(x_k)[\mu(C_k) - \mu(C_{k-1})]$, from which $\nu(C_{m_0}) = \text{(MC)} \int_{C_{m_0}} f d\mu$. It is clear that the strict inequalities are needed to avoid zero in the denominator. This idea is summarized in the following proposition.

Proposition 11. *Consider a maximal chain $\mathcal{C}$ and suppose that $0 < \mu(C_1) < \mu(C_2) < \cdots < \mu(C_m) = \mu(X)$. Then, there exists $f : X \to R^+$ such that*

$$\nu(A) = \text{(MC)} \int_A f d\mu$$

for all $A \in \mathcal{C}$.

Let us denote the function f above as $f = f(\mu, \nu, \mathcal{C})$. This function can be considered as the corresponding derivative.

Definition 10. *Let $\mathcal{C}$ be an arbitrary maximal chain on the reference set X, μ and ν fuzzy measures such that $0 < \mu(C_1) < \mu(C_2) < \cdots < \mu(C_m) = \mu(X)$ for every $\mathcal{C}$. Then, the derivative of ν with respect to μ connected to the Maximal chain-based integral is defined as*

$$\frac{d\nu}{d\mu} := \{f | f = f(\mu, \nu, \mathcal{C}), \mathcal{C} \in \mathcal{M}(X)\}.$$

While taking the Lebesgue and Choquet integral, the derivative is always one function (if it exists), for the Maximal chain-based integral we have the set of derivatives, one for every possible maximal chain. Thus, the number of derivatives is bounded as $\left|\frac{d\nu}{d\mu}\right| \leq |\mathcal{M}(X)| = m!$. Inequality is assumed because not all the derivatives need to exist.

Proposition 12. *Let $\mathcal{C}$ be a maximal chain $\mathcal{C} : \emptyset = C_0 \subset C_1 \subset \cdots \subset C_m = X$ such that $0 < \mu(C_1) < \mu(C_2) < \cdots < \mu(C_m) = \mu(X)$. Suppose that*

$$\frac{(\nu(C_k) - \nu(C_{k-1}))}{(\mu(C_k) - \mu(C_{k-1}))} \geq \frac{\nu(C_{k+1}) - \nu(C_k))}{(\mu(C_{k+1}) - \mu(C_k))}$$

for all $k = 1, 2, \ldots, m-1$. Then

$$\nu(A) = (C) \int_A f d\mu$$

for any $A \in X$, with the Choquet integral on the right.

This is one of the condition for the existence of the Choquet derivative, as proved in [9]. It works because when setting

$$f(x_k) := \frac{(\nu(C_k) - \nu(C_{k-1}))}{(\mu(C_k) - \mu(C_{k-1}))},$$

it implies that $f(x_1) \geq f(x_2) \geq \cdots \geq f(x_m)$, so introducing the order as is needed for the Choquet integral.

Proposition 13. *Let us consider that for every maximal chain in $\mathcal{M}(X)$, the two measures μ and ν are such that $0 < \mu(C_1) < \mu(C_2) < \cdots < \mu(C_m) = \mu(X)$ and $0 < \nu(C_1) < \nu(C_2) < \cdots < \nu(C_m) = \nu(X)$ and let $\frac{d\nu}{d\mu} = \{f_k | k = 1, 2, \cdots, m!\}$. Then we have*

$$\frac{d\mu}{d\nu} = \left(\frac{d\nu}{d\mu}\right)^{-1} = \left\{\frac{1}{f_k} | k = 1, 2, \cdots, m!\right\}.$$

We will now present an example that illustrates these results.

Example 3. *Let us consider two fuzzy measures μ, ν with the values given in the table below.*

	x_1	x_2	x_3	x_1, x_2	x_2, x_3	x_1, x_3	x_1, x_2, x_3
μ	1	2	3	4	5	6	7
ν	3	5	7	6	8	12	15

Then, we find a derivative $f_1 = \frac{d\nu}{d\mu}$ for a chain $\mathcal{C}_1 = \{\emptyset, \{x_1\}, \{x_1, x_2\}, \{x_1, x_2, x_3\}, X\}$.

- *If $A = \{x_1\}$, then $f_1(x_1) = 3$ because*
$$\nu(\{x_1\}) = f_1(x_1)\mu(\{x_1\}).$$
- *If $A = \{x_1, x_2\}$, then*
$$\nu(\{x_1, x_2\}) = f_1(x_1)\mu(\{x_1\}) + f_1(x_2)[\mu(\{x_1, x_2\}) - \mu(\{x_1\})].$$
From that the resulting value is $f_1(x_2) = 1$.
- *If $A = \{x_1, x_2, x_3\}$, then*
$$\begin{aligned}\nu(\{x_1, x_2, x_3\}) = & f_1(x_1)\mu(\{x_1\}) \\ & + f_1(x_2)[\mu(\{x_1, x_2\}) - \mu(\{x_1\})] \\ & + f_1(x_3)[\mu(\{x_1, x_2, x_3\}) - \mu(\{x_1, x_2\})],\end{aligned}$$
so $f_1(x_3) = 3$.

In the same way, other derivatives for all the possible maximal chains can be computed. The resulting derivative *values are listed in the table below. In general, there are six possible derivatives, so*

$$\frac{d\nu}{d\mu} = \{f_1, f_2, f_3, f_4, f_5, f_6\}.$$

Chain			*Derivative*	x_1	x_2	x_3
x_1	x_1, x_2	x_1, x_2, x_3	f_1	3	1	3
x_1	x_1, x_3	x_1, x_2, x_3	f_2	3	$9/5$	3
x_2	x_1, x_2	x_1, x_2, x_3	f_3	5	$1/2$	$9/2$
x_2	x_2, x_3	x_1, x_2, x_3	f_4	5	1	$7/2$
x_3	x_1, x_3	x_1, x_2, x_3	f_5	7	$5/3$	3
x_3	x_2, x_3	x_1, x_2, x_3	f_6	7	$1/2$	$7/2$

Now consider the inverse derivative $\frac{d\mu}{d\nu}$*, first assuming the same chain* C_1 *as before.*

- *If* $A = \{x_1\}$*, then* $\mu(\{x_1\}) = g_1(x_1)\nu(\{x_1\})$*, so* $g_1(x_1) = 1/3$.
- *If* $A = \{x_1, x_2\}$*, then*

  $$\begin{aligned}&\mu(\{x_1, x_2\})\\ &\quad = g_1(x_1)\nu(\{x_1\}) + g_1(x_2)[\nu(\{x_1, x_2\}) - \nu(\{x_1\})]\end{aligned}$$

  and from that $g_1(x_2) = 1$.
- *If* $A = \{x_1, x_2, x_3\}$*, then*

  $$\begin{aligned}\mu(\{x_1, x_2, x_3\}) = {} & g_1(x_1)\nu(\{x_1\})\\ & + g_1(x_2)[\nu(\{x_1, x_2\}) - \nu(\{x_1\})]\\ & + g_1(x_3)[\nu(\{x_1, x_2, x_3\}) - \nu(\{x_1, x_2\})],\end{aligned}$$

  resulting in $g_1(x_3) = 1/3$.

Therefore we have $g_1 = \frac{1}{f_1}$*. In the same way it can be shown that this holds for all the derivatives, thus*

$$\frac{d\mu}{d\nu} = \left\{\frac{1}{f_1}, \frac{1}{f_2}, \frac{1}{f_3}, \frac{1}{f_4}, \frac{1}{f_5}, \frac{1}{f_6}\right\}$$

as proposed in Proposition 13.

IV. Conclusions

The focus of this paper was on the set-valued Maximal chain-based integral. By construction, it is additive for a given chain and can be illustrated through a multi-layer network. We also looked at its special cases obtained by considering particular measures as well as its generalizations.

The corresponding Maximal chain-based derivatives, viewed as a generalization of additive Radon-Nikodym derivatives, were also presented. We provided their definition, which implies that the set of derivatives is always obtained, with each derivative corresponding to one maximal chain. Some basic properties were studied and illustrated by means of an example.

Future research directions include analysis of computational implications of having potentially factorial many maximal chains, thus also the corresponding derivatives, and the characterizations of families of measures that permit us to bound this number in the set-valued integral.

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References

[1] Aumann, R. J. (1965) Integrals of set-valued functions. Journal of mathematical analysis and applications 12.1: 1-12.

[2] Choquet, G. (1953/54) Theory of capacities. Annales de l'institut Fourier, 5 131-295.

[3] Graf, S. (1980) A Radon-Nikodym theorem for capacities. Journal für die reine und angewandte Mathematik, 192-214.

[4] Hardy, G.H., Littlewood, J.E., Pólya, G. (1952), Inequalities. Cambridge: Cambridge University Press.

[5] Hutník, O., Kleinová, M. (2024) Maximal chain-based Choquet-like integrals. Information Sciences 654 119874.

[6] Islam, M., Anderson, D. T., Pinar, A. J., Havens, T. C., Scott, G., Keller, J. M. (2020) Enabling explainable fusion in deep learning with fuzzy integral neural networks. IEEE Transactions on Fuzzy Systems 28:7 1291-1300, July 2020.

[7] Grabisch, M. (1997) k-order additive discrete fuzzy measures and their representation. Fuzzy Sets and Systems 92 167–189.

[8] Mesiar, R. and Šipoš, J. (1993) Radon-Nikodym-like theorem for fuzzy measures. Journal of Fuzzy Mathematics 1, 873–878.

[9] Narukawa, Y., Torra, V. (2019) Derivative for discrete Choquet integrals. In: International Conference on Modeling Decisions for Artificial Intelligence. Springer, 2019. 138-147.

[10] Ontkovičová, Z., and Kiseľák, J. (2022) A way to proper generalization of ϕ-divergence based on Choquet derivatives. Soft Computing 26(21), 11295-11314.

[11] Ontkovičová, Z., and Torra, V. (2025). On measures resulting from the Choquet integral. In: Information Processing and Management of Uncertainty in Knowledge-Based Systems 2, 3-11.

[12] Pereira Dimuro, G., Fernández, J., Bedregal, B. R. C., Mesiar, R., Sanz, J. A., Lucca, G., Bustince, H. (2020) The state-of-art of the generalizations of the Choquet integral: From aggregation and pre-aggregation to ordered directionally monotone functions. Information Fusion 57: 27-43.

[13] Pereira Dimuro, G., Lucca, G., Bedregal, B. R. C., Mesiar, R., Sanz, J. A., Lin, C.-T., Bustince, H. (2020) Generalized CF1F2-integrals: From Choquet-like aggregation to ordered directionally monotone functions. Fuzzy Sets and Systems 378 44-67.

[14] Rébillé, R. (2013) A Super Radon-Nikodym derivative for almost subadditive set functions. International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 21:03, 347-365.

[15] Sugeno, M. (1974) Theory of fuzzy integrals and its applications. Doctoral Thesis, Tokyo Institute of Technology.

[16] Sugeno, M. (2013) A note on derivatives of functions with respect to fuzzy measures. Fuzzy Sets and Systems 222, 1-17.

[17] Torra, V., Narukawa, Y., Sugeno, M. (eds.) (2013) Non-additive measures: theory and applications, Springer.

[18] Wang, Z., and Klir, G. J. (2010) Generalized measure theory (Vol. 25). Springer Science & Business Media.