Activation-Based Grouping and Overlap Operators: Construction and Properties

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***Abstract*—Grouping and overlap operators are widely exploited in aggregation theory and information fusion to model disjunctive and conjunctive characteristics among the data inputs. In this study, we propose a new class of fusion operators, termed as the activation-based grouping and overlap operators. These operators are framed by introducing a threshold-based activation mechanism into the classical ideology of general grouping and overlap operators. The threshold parameter $\tilde{\tau}$ controls the point at which the activation mechanism becomes functional in the aggregation process and is chosen according to the necessities of the decision system. The main idea of the proposed framework is to retain the original behavior of the grouping or overlap operator in the non-activated region, while modifying the response of output near the boundary values by using a suitable activation function. Several construction methods for such operators are discussed, and their mathematical properties are examined. The class of activation-based overlap operators have been studied as a dual class of activation-based grouping operators. The proposed framework offers a flexible way to design a class of fusion operators that are sensitive towards the threshold parameter chosen close to the boundary points, and can be efficiently adapted to different decision scenarios. Due to their adaptive mechanism, such operators can be utilized in a wide range of applications, including decision-making, information fusion, classification problems, and reliability analysis.**

***Index Terms*—Aggregation operators, Grouping and overlap operators, Activation-based grouping and overlap operators, Information fusion.**

I. Introduction

Grouping and overlap operators [1] are nonlinear aggregation operators, which are especially designed to model dependency structure among the inputs, and have been extensively employed in aggregation theory [2], [3], machine learning [4], [5], and information fusion [6]. The general class of grouping operators are formulated to mathematically quantify the degree of support for an entity by aggregating evidence from multiple sources, while emphasizing on the disjunctive behavior among the inputs. In contrast, overlap operators [7] quantify the degree to which multiple criteria simultaneously satisfy a given condition. By explicitly modeling collective interactions among data inputs, these operators have applications in classification problems [8], [9], ensemble learning [10], [11], pattern recognition [12], and reliability analysis [13].

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Despite the widespread utilization of classical grouping and overlap operators, such operators exhibit fixed behavior across the entire input domain, which may limit their suitability in decision scenarios requiring enhanced sensitivity near critical or boundary regions. In many real-world problems, small variation around certain decision thresholds can significantly influence the final outcome, while variation outside this critical regions may be less relevant. For instance, in task of edge detection, intensities of the pixels are selectively enhanced only when the magnitude of the gradient exceeds a predefined threshold, while insignificant fluctuations are effectively suppressed [14], [15]. This enhancement behavior of pixel intensities, is commonly referred as thresholding, and is widely utilized in edge detection. This ideology aids in accurately detecting the meaningful edges [16], [17]. Classical aggregation operators [7], [18] are not designed to explicitly capture such threshold-dependent behavior among data inputs. Motivated by this limitation, we introduce the concept of activation-based grouping and overlap operators. The central idea is to incorporate a threshold-dependent activation mechanism that selectively modifies the response of the operator only in regions where increased sensitivity is required, while preserving the original aggregation behavior elsewhere outside the critical region. This framework allows the aggregation process to reflect the necessities of the decision-system, by adding threshold based enhancement of the response near the critical region without altering the fundamental structure of the underlying grouping or overlap operator. As a result, the proposed operators offer greater flexibility and control in modeling the requirements and behavior of vivid decision-systems, particularly in the applicatory field where decision boundaries or critical regions are of primary importance.

In this study, we develop a general theoretical framework for activation-based grouping and overlap operators. Based on classical theory of grouping and overlap operators, we develop a threshold-based activation function, regulated by the parameter $\tilde{\tau}$, that modifies the aggregation behavior near the boundary region. These aggregation frameworks are designed to ensure that the resulting operators satisfy the significant properties required for grouping and overlap operators in the non-activated regions, and improve the output response obtained from the grouping operator in the activated region chosen near the boundary points. Different choice of activation functions influence the behavior of the proposed operators near the boundary points, particularly in the region close to the activation threshold $\tilde{\tau}$. The proposed framework is flexible and allows the construction of operators with different sensitivity levels, thus makes it possible to adjust the aggregation behavior according to the necessities of the considered decision-system.

The remainder of the paper is organized as follows. Section II briefly reviews the necessary preliminaries on grouping and overlap operators. In Section III, the construction methodology of activation-based grouping operators is presented along with study of several prominent properties. Section IV provides the idea of activation-based overlap operators. Finally, Section V concludes the paper and outlines possible directions for future research.

II. Preliminaries

This section provides a recapitulation for the notion of grouping and overlap operators.

Definition 1 ([19], [20]). *A function $\boldsymbol{F} : [\mathbf{0},\mathbf{1}]^n \to [\mathbf{0},\mathbf{1}]$ is termed as an aggregation operator if:*

1) *$\boldsymbol{F}$ is non-decreasing;*
2) *$\boldsymbol{F}(\tilde{0}) = 0$ and $\boldsymbol{F}(\tilde{1}) = 1$, where $\tilde{0} = (0,\dots,0)$ and $\tilde{1} = (1,\dots,1)$;*

Definition 2 ([20]). *An n-dimensional operator $\boldsymbol{G} : [\mathbf{0},\mathbf{1}]^n \to [\mathbf{0},\mathbf{1}]$ is termed as grouping operator if:*

1) *$\boldsymbol{G}$ is a symmetric operator;*
2) *$\boldsymbol{G}$ is monotonically increasing;*
3) *$\boldsymbol{G}$ is continuous;*
4) *$\boldsymbol{G}(\tilde{z}) = 1 \iff$ for $\tilde{z} = (z_1,\dots,z_n)$, $\exists\, z_i : z_i = 1$;*
5) *$\boldsymbol{G}(\tilde{z}) = 0 \iff$ for $\tilde{z} = (z_1,\dots,z_n)$, $z_i = 0\ \forall\, i$.*

Example II.1. *Let $\tilde{z} = (z_1,\dots,z_n) \in [\mathbf{0},\mathbf{1}]^{\mathbf{n}}$. Then, examples of grouping operators are given below:*

1) $\boldsymbol{G}(\tilde{z})=\frac{\max\limits_i(z_i)}{\max\limits_i(z_i)+\sqrt{\prod\limits_i(1-z_i)}}$.
2) $\boldsymbol{G}(\tilde{z}) = 1 - \prod\limits_i(1 - z_i)$.
3) $\boldsymbol{G}(\tilde{z}) = \max\limits_i(z_i)^r$ *with* $r > 0$.
4) $\boldsymbol{G}(\tilde{z}) = 1 - \sqrt{\prod\limits_i(1 - z_i)}$.

Definition 3 ([1], [20]). *An n-dimensional operator $\boldsymbol{O} : [\mathbf{0},\mathbf{1}]^n \to [\mathbf{0},\mathbf{1}]$ is termed as an overlap operator if:*

1) *$\boldsymbol{O}$ is a symmetric operator;*
2) *$\boldsymbol{O}$ is monotonically increasing;*
3) *$\boldsymbol{O}$ is continuous;*
4) *$\boldsymbol{O}(\tilde{z}) = 1 \iff$ for $\tilde{z} = (z_1,\dots,z_n)$, $z_i = 1\ \forall\, i$;*
5) *$\boldsymbol{O}(\tilde{z}) = 0 \iff$ for $\tilde{z} = (z_1,\dots,z_n)$ $\exists\, z_i : z_i = 0$.*

Example II.2. *Examples of overlap operators are :*

1) $\boldsymbol{O}(\tilde{z}) = \sin\frac{\pi}{2}(\prod\limits_i z_i)^r$ *with* $r > 0$.
2) $\boldsymbol{O}(\tilde{z}) = \prod\limits_i(z_i)$.
3) $\boldsymbol{O}(\tilde{z}) = \min\limits_i(z_i)^r\ r > 0$.
4) $\boldsymbol{O}(\tilde{z})=\frac{\prod\limits_i(z_i)}{1+\prod\limits_i(1-z_i)}$.

Definition 4 ([21]). *A mapping $G_O : [0,1]^n \to [0,1]$ is called a general overlap operator if, for any vector $\tilde{z} = (z_1, z_2, \dots, z_n) \in [0,1]^n$, the following conditions are satisfied:*

1) *G_O is symmetric.*
2) *If $\prod_{i=1}^n z_i = 0$, then $G_O(\tilde{z}) = 0$.*
3) *If $\prod_{i=1}^n z_i = 1$, then $G_O(\tilde{z}) = 1$.*
4) *G_O is increasing in each argument.*
5) *G_O is continuous on $[0,1]^n$.*

III. Activation based grouping operators

Definition 5. *Let $\boldsymbol{G} : [\mathbf{0},\mathbf{1}]^n \to [\mathbf{0},\mathbf{1}]$ be a grouping operator. Let $\tilde{\tau}$, $\tilde{\beta}$ be real numbers such that $\tilde{\tau} \le \tilde{\beta} \le 1$, where $\tilde{\tau}$ is an activation threshold chosen close to 1. Let $\boldsymbol{s_G} : [\tilde{\tau}, 1] \to [\tilde{\beta}, 1]$ be a monotonically increasing function satisfying $\boldsymbol{s_G}(1) = 1$. The function $\boldsymbol{G^*_{\tilde{\tau}}}$:$[\mathbf{0},\mathbf{1}]^n \to [\mathbf{0},\mathbf{1}]$ defined by:*

$$\boldsymbol{G^*_{\tilde{\tau}}}(\tilde{z}) = \begin{cases} \boldsymbol{G}(\tilde{z}), & if\ \boldsymbol{G}(\tilde{z}) < \tilde{\tau}, \\ \boldsymbol{s_G}(\boldsymbol{G}(\tilde{z}))\,, & if\ \boldsymbol{G}(\tilde{z}) \ge \tilde{\tau}, \end{cases}$$

is termed as an activation-based grouping operator, generated by the grouping operator $\boldsymbol{G}$, and the activation threshold $\tilde{\tau}$. The function $\boldsymbol{s_G}$ is referred as an activation function for the operator $\boldsymbol{G^*_{\tilde{\tau}}}$.

Remark 1. *The expression that the activation threshold $\tilde{\tau}$ is 'close to 1' can be interpreted in a quantitative manner. In particular, one may consider $\tilde{\tau} = 1-\varepsilon$, where $\varepsilon > 0$ is a small parameter. The choice of ε is not universal and depends on the characteristics of the underlying decision system, such as the level of sensitivity required, and the tolerance for deviation from full agreement.*

Classical grouping operators exhibit a rigid boundary condition by requiring that the output value will attain the unit value whenever at least one input argument equals one. In practical settings, such a requirement may be strict in nature, since values close to one already indicate a strong contribution to the aggregated outcome. The operator defined above replaces this crisp condition with a threshold-based criteria, allowing greater tolerance to small variations while retaining the essential characteristics of grouping operators.

Remark 2. *From the above definition, the following observations can be made:*

(i) *The function $\boldsymbol{G^*_{\tilde{\tau}}}$ is symmetric, since, it is exploited by a symmetric operator $\boldsymbol{G}$.*
(ii) *The activation-based grouping operator $\boldsymbol{G^*_{\tilde{\tau}}}$ is a monotonically increasing operator since, the activation function $\boldsymbol{s_G}$ is a monotonically increasing function.*
(iii) *If $\boldsymbol{G}(\tilde{z}) < \tilde{\tau}$, then $\boldsymbol{G^*_{\tilde{\tau}}}(\tilde{z}) = \boldsymbol{G}(\tilde{z})$, and hence, no activation is required within this region. If $\boldsymbol{G}(\tilde{z}) \ge \tilde{\tau}$, then the activation function is utilized to improve the response of $\boldsymbol{G}$, and then $\boldsymbol{G^*_{\tilde{\tau}}}(\tilde{z}) = \boldsymbol{s_G}(\boldsymbol{G}(\tilde{z})) \in [\tilde{\beta}, 1]$.*

(iv) *The function $\boldsymbol{G}^*_{\tilde{\tau}}$ need not be continuous. For instance, Consider the activation-based grouping operator defined as follows:*

$$\boldsymbol{G}^*_{\tilde{\tau}}(\tilde{z}) = \begin{cases} \boldsymbol{G}(\tilde{z}), & if\ \boldsymbol{G}(\tilde{z}) < \tilde{\tau}, \\ 1, & if\ \boldsymbol{G}(\tilde{z}) \geq \tilde{\tau}, \end{cases}$$

where $\boldsymbol{G}$ is any grouping operator and the activation function is taken as $\boldsymbol{s_G}(t) = 1\ \forall\ t \in [\tilde{\tau}, 1]$. Then clearly $\boldsymbol{G}^*_{\tilde{\tau}}$ is not continuous.

(v) $\boldsymbol{G}^*_{\tilde{\tau}}(\tilde{z}) = 0 \iff z_i = 0\ \forall\ i.$

Remark 3. *On choosing $\tilde{\tau} = 1$ in the activation based grouping operator, and considering that $\boldsymbol{s_G}(1) = 1$, we have $\boldsymbol{G}^*_{\tilde{\tau}}(\tilde{z}) = \boldsymbol{G}(\tilde{z})\ \forall\ \tilde{z} \in [0,1]^n$. Thus, the activation-based grouping operator reduces to the classical grouping operator. Furthermore, if $\tilde{\tau} < 1$ and the activation function is chosen as $\boldsymbol{s_G}(t) = 1$ for all $t \in [\tilde{\tau}, 1]$, then once the activation threshold is reached, the aggregation operator value jumps immediately to 1.*

Example III.1. 1) *On considering the activation function*

$$\boldsymbol{s}_1(t) = \tilde{\beta} + \frac{1-\tilde{\beta}}{1-\tilde{\tau}}\,(t-\tilde{\tau}), \quad t \in [\tilde{\tau}, 1].$$

and the grouping operator $\boldsymbol{G}(\tilde{z})$, defined as $\boldsymbol{G}(\tilde{z}) = \max_i(z_i)$, the following activation-based grouping operator is obtained:

$$\boldsymbol{G}^*_{\tilde{\tau}}(\tilde{z}) = \begin{cases} \max_i(z_i), & \boldsymbol{G}(\tilde{z}) < \tilde{\tau}, \\ \tilde{\beta} + \frac{1-\tilde{\beta}}{1-\tilde{\tau}}\,(\max_i(z_i) - \tilde{\tau}), & \boldsymbol{G}(\tilde{z}) \geq \tilde{\tau}. \end{cases}$$

2) *On considering the activation function*

$$\boldsymbol{s}_2(t) = \tilde{\beta} + (1-\tilde{\beta})\left(\frac{t-\tilde{\tau}}{1-\tilde{\tau}}\right)^{\alpha}, \quad t \in [\tilde{\tau}, 1],\ \alpha > 1.$$

and the grouping operator $\boldsymbol{G}(\tilde{z})$, defined as $\boldsymbol{G}(\tilde{z}) = 1 - \prod_i(1-z_i)$, the following activation-based grouping operator is obtained:

$$\boldsymbol{G}^*_{\tilde{\tau}}(\tilde{z}) = \begin{cases} 1 - \prod_i(1-z_i), & \boldsymbol{G}(\tilde{z}) < \tilde{\tau} \\ \tilde{\beta} + (1-\tilde{\beta})\left(\frac{1-\prod_i(1-z_i)-\tilde{\tau}}{1-\tilde{\tau}}\right)^{\alpha}, & \boldsymbol{G}(\tilde{z}) \geq \tilde{\tau}. \end{cases}$$

3) *On considering the activation function*

$$\boldsymbol{s}_3(t) = \tilde{\beta} + (1-\tilde{\beta})\,\frac{e^{\lambda(t-\tilde{\tau})}-1}{e^{\lambda(1-\tilde{\tau})}-1}, \quad t \in [\tilde{\tau}, 1],\ \lambda > 0.$$

and the grouping operator $\boldsymbol{G}(\tilde{z})$, defined as $\boldsymbol{G}(\tilde{z}) = 1 - \sqrt{\prod_i(1-z_i)}$, the following activation-based grouping operator is obtained

$$\boldsymbol{G}^*_{\tilde{\tau}}(\tilde{z}) = \begin{cases} 1 - \sqrt{\prod_i(1-z_i)}, & \boldsymbol{G}(\tilde{z}) < \tilde{\tau} \\ \tilde{\beta} + (1-\tilde{\beta})\,\frac{e^{\lambda\left(1-\sqrt{\prod_i(1-z_i)}-\tilde{\tau}\right)}-1}{e^{\lambda(1-\tilde{\tau})}-1}, & \boldsymbol{G}(\tilde{z}) \geq \tilde{\tau}. \end{cases}$$

4) *On considering the activation function*

$$\boldsymbol{s}_4(t) = 1, \quad t \in [\tilde{\tau}, 1],$$

and the grouping operator $\boldsymbol{G}(\tilde{z})$, defined as $\boldsymbol{G}(\tilde{z}) = \frac{\max_i(z_i)}{\max_i(z_i)+\sqrt{\prod_i(1-z_i)}}$, the following activation-based grouping operator is obtained:

$$\boldsymbol{G}^*_{\tilde{\tau}}(\tilde{z}) = \begin{cases} \frac{\max_i(z_i)}{\max_i(z_i)+\sqrt{\prod_i(1-z_i)}} & \boldsymbol{G}(\tilde{z}) < \tilde{\tau} \\ 1 & \boldsymbol{G}(\tilde{z}) \geq \tilde{\tau} \end{cases}$$

Theorem III.1. *Let $\boldsymbol{G} : [\mathbf{0},\mathbf{1}]^n \to [\mathbf{0},\mathbf{1}]$ be a grouping operator and let $\tilde{\tau}$ be a fixed threshold such that $\tilde{\tau} \leq 1$. Let $\boldsymbol{G}^*_{\mathbf{1},\tilde{\tau}}$ and $\boldsymbol{G}^*_{\mathbf{2},\tilde{\tau}}$ be two activation-based grouping operators constructed from $\boldsymbol{G}$ using the same threshold $\tilde{\tau}$ and activation functions $\boldsymbol{s_{G_1}} : [\tilde{\tau}, 1] \to [\tilde{\beta}_1, 1]$, $\boldsymbol{s_{G_2}} : [\tilde{\tau}, 1] \to [\tilde{\beta}_2, 1]$, respectively, where $\tilde{\tau} \leq \tilde{\beta}_1, \tilde{\beta}_2 \leq 1$ and $\boldsymbol{s_{G_1}}(1) = \boldsymbol{s_{G_2}}(1) = 1$. Define a function $\boldsymbol{H}^*_{\tilde{\tau}} : [\mathbf{0},\mathbf{1}]^n \to [\mathbf{0},\mathbf{1}]$, by:*

$$\boldsymbol{H}^*_{\tilde{\tau}}(\tilde{z}) = \begin{cases} \boldsymbol{G}(\tilde{z}), & \text{if } \boldsymbol{G}(\tilde{z}) < \tilde{\tau}, \\ \boldsymbol{s_{G_2}}\big(\boldsymbol{s_{G_1}}(\boldsymbol{G}(\tilde{z}))\big), & \text{if } \boldsymbol{G}(\tilde{z}) \geq \tilde{\tau}. \end{cases}$$

Then $\boldsymbol{H}^*_{\tilde{\tau}}$ is an activation-based grouping operator.

Proof. Since, $\boldsymbol{s_{G_1}}$ and $\boldsymbol{s_{G_2}}$ are both activation functions, both of them are monotonically increasing real valued functions and moreover, $\boldsymbol{s_{G_1}}(1) = \boldsymbol{s_{G_2}}(1) = 1$. Now, for any $t_1, t_2 \in [\tilde{\tau}, 1]$ with $t_1 \leq t_2$ we will have $\boldsymbol{s_{G_1}}(t_1) \leq \boldsymbol{s_{G_1}}(t_2)$ and consequently, $(\boldsymbol{s_{G_2}} \circ \boldsymbol{s_{G_1}})(t_1) \leq (\boldsymbol{s_{G_2}} \circ \boldsymbol{s_{G_1}})(t_2)$ and hence, $\boldsymbol{s_{G_2}} \circ \boldsymbol{s_{G_1}} : [\tilde{\tau}, 1] \to [\tilde{\beta}_2, 1]$, is also an increasing real valued function, and $(\boldsymbol{s_{G_2}} \circ \boldsymbol{s_{G_1}})(1) = 1$, so it is clear that $\boldsymbol{H}^*_{\tilde{\tau}}$ is an activation-based grouping operator with activation function $\boldsymbol{s_{G_2}} \circ \boldsymbol{s_{G_1}}$ and with activation threshold $\tilde{\tau}$. □

Example III.2. *Let $\boldsymbol{s_{G_1}} : [\tilde{\tau}, 1] \to [\tilde{\beta}_1, 1]$ and $\boldsymbol{s_{G_2}} : [\tilde{\tau}, 1] \to [\tilde{\beta}_2, 1]$ given by, $\boldsymbol{s_{G_1}}(t) = \tilde{\beta}_1 + \frac{1-\tilde{\beta}_1}{1-\tilde{\tau}}\,(t-\tilde{\tau})$ and $\boldsymbol{s_{G_2}}(t) = \tilde{\beta}_2 + (1-\tilde{\beta}_2)(\frac{t-\tilde{\tau}}{1-\tilde{\tau}})^{\alpha}$, where $\alpha > 1$, be two activation functions. Let $\boldsymbol{G}(\tilde{z}) = \max_i(z_i)$ be the grouping operator where $\tilde{z} = (z_1, \ldots, z_n)$. Let $\boldsymbol{G}^*_{\mathbf{1},\tilde{\tau}}$ and $\boldsymbol{G}^*_{\mathbf{2},\tilde{\tau}}$ are two activation-based grouping operators which are generated from $\boldsymbol{G}$ using $\boldsymbol{s_{G_1}}$ and $\boldsymbol{s_{G_2}}$, respectively. Then, using the composition theorem as stated above we can form a new activation-based grouping operator $\boldsymbol{H}^*_{\tilde{\tau}} : [\mathbf{0},\mathbf{1}]^n \to [\mathbf{0},\mathbf{1}]$, given by:*

$$\boldsymbol{H}^*_{\tilde{\tau}}(\tilde{z}) = \begin{cases} \max_i z_i, & \text{if } \boldsymbol{G}(\tilde{z}) < \tilde{\tau}, \\ \boldsymbol{s_{G_2}}\big(\boldsymbol{s_{G_1}}(\max_i z_i)\big), & \text{if } \boldsymbol{G}(\tilde{z}) \geq \tilde{\tau}. \end{cases}$$

Remark 4. *Let $\boldsymbol{G}^*_{\mathbf{1},\tilde{\tau}}$ and $\boldsymbol{G}^*_{\mathbf{2},\tilde{\tau}}$ be two activation-based grouping operators on $[\mathbf{0},\mathbf{1}]^n$, constructed with the same activation threshold $\tilde{\tau}$ and the grouping operator $\boldsymbol{G}$, with activation functions $\boldsymbol{s_{G_1}} : [\tilde{\tau}, 1] \to [\tilde{\beta}_1, 1]$ and $\boldsymbol{s_{G_2}} : [\tilde{\tau}, 1] \to [\tilde{\beta}_2, 1]$ respectively, where $\tilde{\tau} \leq \tilde{\beta}_1, \tilde{\beta}_2 \leq 1$ and $\boldsymbol{s_{G_1}}(1) = \boldsymbol{s_{G_2}}(1) = 1$. Define the function $\boldsymbol{H}^*_{\tilde{\tau}} : [\mathbf{0},\mathbf{1}]^n \to [\mathbf{0},\mathbf{1}]$ by:*

$$\boldsymbol{H}^*_{\tilde{\tau}}(\tilde{z}) = \max\left\{\boldsymbol{G}^*_{\mathbf{1},\tilde{\tau}}(\tilde{z}),\ \boldsymbol{G}^*_{\mathbf{2},\tilde{\tau}}(\tilde{z})\right\}.$$

Then, $\boldsymbol{H}^*_{\tilde{\tau}}$ is an activation-based grouping operator.

Proof. Since, we know that s_{G_1} and s_{G_2} are both activation functions, so $s_{G_1}(1) = 1$ and $s_{G_2}(1) = 1$, and both s_{G_1} and s_{G_2} are monotonically increasing operators. Hence, $\max\{s_{G_1}, s_{G_2}\}$ is also monotonically increasing operator where $\max\{s_{G_1}, s_{G_2}\} : [\tilde{\tau}, 1] \to [\tilde{\beta}_2, 1]$ (or $[\tilde{\tau}, 1] \to [\tilde{\beta}_1, 1]$). Hence, $\boldsymbol{H}^*_{\tilde{\tau}}$ is an activation-based grouping operator with activation threshold $\tilde{\tau}$ and with activation function $\max\{s_{G_1}, s_{G_2}\}$. $\square$

Example III.3. *Let $s_{G_1} : [\tilde{\tau}, 1] \to [\tilde{\beta}_1, 1]$ and $s_{G_2} : [\tilde{\tau}, 1] \to [\tilde{\beta}_2, 1]$, given by, $s_{G_1}(t) = \tilde{\beta}_1 + \frac{1-\tilde{\beta}_1}{1-\tilde{\tau}}(t - \tilde{\tau})$ and $s_{G_2}(t) = \tilde{\beta}_2 + (1-\tilde{\beta}_2)(\frac{t-\tilde{\tau}}{1-\tilde{\tau}})^{\alpha}$ where $\alpha > 1$, respectively. Let us consider the grouping operator $\boldsymbol{G}(\tilde{z})$, defined as $\boldsymbol{G}(\tilde{z}) = 1 - \prod_i (1 - z_i)$. Then, from the above theorem we can obtain the activation-based grouping operator $\boldsymbol{H}^*_{\tilde{\tau}} : [\mathbf{0}, \mathbf{1}]^n \to [\mathbf{0}, \mathbf{1}]$ given by*

$$\boldsymbol{H}^*_{\tilde{\tau}}(\tilde{z}) = \max\left\{\boldsymbol{G}^*_{\mathbf{1},\tilde{\tau}}(\tilde{z}),\, \boldsymbol{G}^*_{\mathbf{2},\tilde{\tau}}(\tilde{z})\right\},$$

where the functions $\boldsymbol{G}^*_{\mathbf{1},\tilde{\tau}}$ and $\boldsymbol{G}^*_{\mathbf{2},\tilde{\tau}}$ are defined as:

$$\boldsymbol{G}^*_{\mathbf{1},\tilde{\tau}}(\tilde{z}) = \begin{cases} 1 - \prod_i (1 - z_i), & \text{if } \boldsymbol{G}(\tilde{z}) < \tilde{\tau}, \\ \tilde{\beta}_1 + \frac{1-\tilde{\beta}_1}{1-\tilde{\tau}}\left(1 - \prod_i (1 - z_i) - \tilde{\tau}\right), & \text{if } \boldsymbol{G}(\tilde{z}) \geq \tilde{\tau}. \end{cases}$$

$$\boldsymbol{G}^*_{\mathbf{2},\tilde{\tau}}(\tilde{z}) = \begin{cases} 1 - \prod_i (1 - z_i), & \text{if } \boldsymbol{G}(\tilde{z}) < \tilde{\tau}, \\ \tilde{\beta}_2 + (1 - \tilde{\beta}_2)\left(\frac{1 - \prod_i (1-z_i) - \tilde{\tau}}{1-\tilde{\tau}}\right)^{\alpha}, & \text{if } \boldsymbol{G}(\tilde{z}) \geq \tilde{\tau}. \end{cases}$$

Theorem III.2. *Let $\boldsymbol{G} : [\mathbf{0}, \mathbf{1}]^n \to [\mathbf{0}, \mathbf{1}]$ be a grouping operator. Let $\boldsymbol{G}^*_{\mathbf{1},\tilde{\tau}}$ and $\boldsymbol{G}^*_{\mathbf{2},\tilde{\tau}}$ be two activation-based grouping operators constructed from the same grouping operator $\boldsymbol{G}$ and the same activation threshold $\tilde{\tau}$, but with possibly different activation functions $s_{G_1} : [\tilde{\tau}, 1] \to [\tilde{\beta}_1, 1]$ and $s_{G_2} : [\tilde{\tau}, 1] \to [\tilde{\beta}_2, 1]$, where $\tilde{\tau} \leq \tilde{\beta}_1, \tilde{\beta}_2 \leq 1$ and $s_{G_1}(1) = s_{G_2}(1) = 1$, respectively. Let $\lambda \in [\mathbf{0}, \mathbf{1}]$ and define the function $\boldsymbol{H}^*_{\tilde{\tau}} : [\mathbf{0}, \mathbf{1}]^n \to [\mathbf{0}, \mathbf{1}]$ by:*

$$\boldsymbol{H}^*_{\tilde{\tau}}(\tilde{z}) = \lambda\, \boldsymbol{G}^*_{\mathbf{1},\tilde{\tau}}(\tilde{z}) + (1 - \lambda)\, \boldsymbol{G}^*_{\mathbf{2},\tilde{\tau}}(\tilde{z}).$$

Then, $\boldsymbol{H}^*_{\tilde{\tau}}$ is an activation based grouping operator.

Proof. Since, we know that s_{G_1} and s_{G_2} are both activation functions, so $s_{G_1}(1) = 1$ and $s_{G_2}(1) = 1$. Also, both s_{G_1} and s_{G_2} are monotonically increasing real valued functions. Now, $s_{G_{H^*_{\tilde{\tau}}}}(t) = \lambda\, s_{G_1}(t) + (1 - \lambda)\, s_{G_2}(t),\ t \in [\tilde{\tau}, 1]$, is a monotonically increasing function as it is the convex combination of two monotonically increasing functions and also $s_{G_{H^*_{\tilde{\tau}}}}(1) = 1$. Then, we can conclude that $\boldsymbol{H}^*_{\tilde{\tau}}$ is an activation-based grouping operator with activation threshold $\tilde{\tau}$ and activation function $s_{G_{H^*_{\tilde{\tau}}}} : [\tilde{\tau}, 1] \to \left[\lambda\tilde{\beta}_1 + (1 - \lambda)\tilde{\beta}_2,\ 1\right]$. $\square$

Theorem III.3. *Let $\boldsymbol{G} : [\mathbf{0}, \mathbf{1}]^n \to [\mathbf{0}, \mathbf{1}]$ be a grouping operator. Let $\boldsymbol{G}^*_{\mathbf{1},\tilde{\tau}}$ and $\boldsymbol{G}^*_{\mathbf{2},\tilde{\tau}}$ be two activation-based grouping operators constructed from the same grouping operator $\boldsymbol{G}$ and the same activation threshold $\tilde{\tau}$, with activation functions s_{G_1} and s_{G_2}, respectively. Assume that $s_{G_1} : [\tilde{\tau}, 1] \to [\tilde{\beta}_1, 1]$ and $s_{G_2} : [\tilde{\tau}, 1] \to [\tilde{\beta}_2, 1]$, where $\tilde{\tau} \leq \tilde{\beta}_1 \leq \tilde{\beta}_2 \leq 1$. If $s_{G_1}(t) \leq s_{G_2}(t)\ \forall\ t \in [\tilde{\tau}, 1]$, then $\boldsymbol{G}^*_{\mathbf{1},\tilde{\tau}}(\tilde{z}) \leq \boldsymbol{G}^*_{\mathbf{2},\tilde{\tau}}(\tilde{z}),\ \forall\ \tilde{z} \in [\mathbf{0}, \mathbf{1}]^n$.*

Proof. If $s_{G_1}(t) \leq s_{G_2}(t),\ \forall\ t \in [\tilde{\tau}, 1]$, then $s_{G_1}(\mathbf{G}(\tilde{z})) \leq s_{G_2}(\mathbf{G}(\tilde{z}))\ \forall\ \tilde{z}$, when $\mathbf{G}(\tilde{z}) \geq \tilde{\tau}$. Since, $\boldsymbol{G}^*_{\mathbf{1},\tilde{\tau}}$ and $\boldsymbol{G}^*_{\mathbf{2},\tilde{\tau}}$ are formed from the same grouping operator $\mathbf{G}$, the result follows immediately from the definition of activation based grouping operators. $\square$

Theorem III.4. *Let $\boldsymbol{G} : [\mathbf{0}, \mathbf{1}]^n \to [\mathbf{0}, \mathbf{1}]$ be a grouping operator. Let $\boldsymbol{G}^*_{\tilde{\tau}}$ be an activation-based grouping operator constructed from $\boldsymbol{G}$ using an activation threshold $\tilde{\tau}$ and an activation function s_G such that it satisfies $s_G \circ s_G = s_G$. Let $(\boldsymbol{G}^*_{\tilde{\tau}})^*$ denote the activation-based grouping operator obtained by applying the same activation mechanism again to $\boldsymbol{G}^*_{\tilde{\tau}}$, using the same threshold $\tilde{\tau}$ and the activation function s_G. Then, $(\boldsymbol{G}^*_{\tilde{\tau}})^* = \boldsymbol{G}^*_{\tilde{\tau}}$.*

Proof. Here the activation function is $s_G : [\tilde{\tau}, 1] \to [\tilde{\beta}, 1]$. Now, using the definition of $(\boldsymbol{G}^*_{\tilde{\tau}})^*$ we can write

$$(\boldsymbol{G}^*_{\tilde{\tau}})^*(\tilde{z}) = \begin{cases} \boldsymbol{G}^*_{\tilde{\tau}}(\tilde{z}), & \text{if } \boldsymbol{G}^*_{\tilde{\tau}}(\tilde{z}) < \tilde{\tau}, \\ s_G\left(\boldsymbol{G}^*_{\tilde{\tau}}(\tilde{z})\right), & \text{if } \boldsymbol{G}^*_{\tilde{\tau}}(\tilde{z}) \geq \tilde{\tau}, \end{cases}$$

where $\tilde{z}$ is an input value from $[\mathbf{0}, \mathbf{1}]^n$. Now, if $\boldsymbol{G}^*_{\tilde{\tau}}(\tilde{z}) < \tilde{\tau}$, then we claim that $\mathbf{G}(\tilde{z}) < \tilde{\tau}$, as otherwise if $\mathbf{G}(\tilde{z}) \geq \tilde{\tau}$, then $\boldsymbol{G}^*_{\tilde{\tau}}(\tilde{z}) = s_G(\mathbf{G}(\tilde{z}))$ and consequently $\boldsymbol{G}^*_{\tilde{\tau}}(\tilde{z}) \geq \tilde{\tau}$, which is a contradiction to our assumption. Hence, if $\boldsymbol{G}^*_{\tilde{\tau}}(\tilde{z}) < \tilde{\tau}$, then $\boldsymbol{G}^*_{\tilde{\tau}}(\tilde{z}) = \mathbf{G}(\tilde{z})$. Similarly, if $\boldsymbol{G}^*_{\tilde{\tau}}(\tilde{z}) \geq \tilde{\tau}$, then $\mathbf{G}(\tilde{z}) \geq \tilde{\tau}$ and then $\boldsymbol{G}^*_{\tilde{\tau}}(\tilde{z}) = s_G(\mathbf{G}(\tilde{z}))$. So, we will get that $(\boldsymbol{G}^*_{\tilde{\tau}})^*(\tilde{z}) = \mathbf{G}(\tilde{z})$ if $\boldsymbol{G}^*_{\tilde{\tau}}(\tilde{z}) < \tilde{\tau}$, and $(\boldsymbol{G}^*_{\tilde{\tau}})^*(\tilde{z}) = s_G(s_G(\mathbf{G}(\tilde{z})))$ if $\boldsymbol{G}^*_{\tilde{\tau}}(\tilde{z}) \geq \tilde{\tau}$. Now, using the fact that s_G is an idempotent function, we can conclude that $(\boldsymbol{G}^*_{\tilde{\tau}})^* = \boldsymbol{G}^*_{\tilde{\tau}}$. $\square$

Let $\boldsymbol{O} : [\mathbf{0}, \mathbf{1}]^n \to [\mathbf{0}, \mathbf{1}]$ be an overlap operator. Then, by [20], we know that $\boldsymbol{O}$ can be expressed as:
$\boldsymbol{O}(\tilde{z}) = \frac{p^*(\tilde{z})}{p^*(\tilde{z}) + q^*(\tilde{z})}$,
for some $p^*, q^* : [\mathbf{0}, \mathbf{1}]^n \to [\mathbf{0}, \mathbf{1}]$, such that:

1) p^* and q^* are symmetric functions;
2) p^* is increasing and q^* is decreasing;
3) $p^*(\tilde{z}) = 0 \iff \prod_i z_i = 0$;
4) $q^*(\tilde{z}) = 0 \iff \prod_i z_i = 1$;
5) p^* and q^* are continuous functions.

Theorem III.5. *Let $\boldsymbol{O} : [\mathbf{0}, \mathbf{1}]^n \to [\mathbf{0}, \mathbf{1}]$ be an overlap operator represented in the form $\boldsymbol{O}(\tilde{z}) = \frac{p^*(\tilde{z})}{p^*(\tilde{z}) + q^*(\tilde{z})}$, where p^* and q^* satisfy the above mentioned properties and let $\boldsymbol{G}(\tilde{z}) = 1 - \boldsymbol{O}(\tilde{1} - \tilde{z})$. Then, $\boldsymbol{G}$ is a grouping operator. Moreover, if we choose real numbers $\tilde{\tau}$, $\tilde{\beta}$ such that $\tilde{\tau} \leq \tilde{\beta}$ and a monotonically increasing real valued function $s_G : [\tilde{\tau}, 1] \to [\tilde{\beta}, 1]$ such that $s_G(1) = 1$, then the function $\boldsymbol{G}^*_{\tilde{\tau}} : [\mathbf{0}, \mathbf{1}]^n \to [\mathbf{0}, \mathbf{1}]$, defined as:*

$$\boldsymbol{G}^*_{\tilde{\tau}}(\tilde{z}) = \begin{cases} \boldsymbol{Q}^*(\tilde{z}), & \boldsymbol{Q}^*(\tilde{z}) < \tilde{\tau}, \\ s_G\left(\boldsymbol{Q}^*(\tilde{z})\right), & \boldsymbol{Q}^*(\tilde{z}) \geq \tilde{\tau} \end{cases}$$

where, $\boldsymbol{Q^*}(\tilde{z}) = 1 - \frac{p^*(\tilde{1}-\tilde{z})}{p^*(\tilde{1}-\tilde{z})+q^*(\tilde{1}-\tilde{z})}$, $\forall \tilde{z}$. *Then,* $\boldsymbol{G^*_{\tilde{\tau}}}$ *is an activation-based grouping operator.*

Proof. It is clear that $\mathbf{G}$ is symmetric and continuous as p^* and q^* are symmetric and continuous. $\mathbf{G}(\tilde{z}) = 1$ if and only if $\exists\, z_i$ such that $z_i = 1$, and $\mathbf{G}(\tilde{z}) = 0$ if and only if $z_i = 0\ \forall\ i$. So, $\mathbf{G}$ is a grouping operator. Now, from the way $\boldsymbol{G^*_{\tilde{\tau}}}$ is defined, it is clear that $\boldsymbol{G^*_{\tilde{\tau}}}$ is an activation-based grouping operator with activation function $\boldsymbol{s_G}$ and activation threshold $\tilde{\tau}$. □

Example III.4. *We take* $p^*(\tilde{z}) = (\prod\limits_i z_i)^{\frac{1}{n}}$ *and* $q^*(\tilde{z}) = \max\limits_i(1-z_i)$. *Then, we construct* $\boldsymbol{O}(\tilde{z}) = \frac{(\prod_i z_i)^{\frac{1}{n}}}{(\prod_i z_i)^{\frac{1}{n}}+\max_i(1-z_i)}$, *and then* $\boldsymbol{G}(\tilde{z})= 1 - \boldsymbol{O}(1-\tilde{z})$*, is a grouping operator. Now, taking the function* $\boldsymbol{s_G}(t) = \tilde{\beta} + \frac{1-\tilde{\beta}}{1-\tilde{\tau}}(t-\tilde{\tau})$, $t \in [\tilde{\tau}, 1]$, *we can form the activation-based grouping operator* $\boldsymbol{G^*_{\tilde{\tau}}}$ *defined as:*

$$\boldsymbol{G^*_{\tilde{\tau}}}(\tilde{z}) = \begin{cases} \boldsymbol{Q^*}(\tilde{z}), & \boldsymbol{Q^*}(\tilde{z}) < \tilde{\tau} \\ \boldsymbol{s_G}\left(\boldsymbol{Q^*}(\tilde{z})\right), & \boldsymbol{Q^*}(\tilde{z}) \geq \tilde{\tau} \end{cases}$$

where, $\boldsymbol{Q^*}(\tilde{z}) = 1 - \frac{\prod_i(1-z_i)^{\frac{1}{n}}}{\prod_i(1-z_i)^{\frac{1}{n}}+\max_i z_i}$, $\forall \tilde{z}$.

Remark 5. *Let* $\boldsymbol{G} : [\mathbf{0},\mathbf{1}]^n \to [\mathbf{0},\mathbf{1}]$ *be a grouping operator. Let* $\boldsymbol{G^*_{\tilde{\tau}}}$ *be an activation-based grouping operator constructed from* $\boldsymbol{G}$ *using an activation threshold* $\tilde{\tau}$ *and an activation function* $\boldsymbol{s_G}$. *If the activation function* $\boldsymbol{s_G}$ *is continuous on* $[\tilde{\tau}, 1]$ *and satisfies* $\boldsymbol{s_G}(\tilde{\tau}) = \tilde{\tau}$, *then* $\boldsymbol{G^*_{\tilde{\tau}}}$ *is a continuous function.*

Proof. Since, the activation function $\boldsymbol{s_G}$ is continuous, then using the fact that composition of two continuous functions is again continuous and $\boldsymbol{s_G}(\tilde{\tau}) = \tilde{\tau}$, we will get that $\boldsymbol{G^*_{\tilde{\tau}}}$ is continuous. □

IV. Activation-based overlap functions

Definition 6. *Let* $\boldsymbol{O} : [\mathbf{0},\mathbf{1}]^n \to [\mathbf{0},\mathbf{1}]$ *be an overlap operator. Let* $\tilde{\tau}$ *and* $\tilde{\beta}$ *be real numbers such that* $0 \leq \tilde{\beta} \leq \tilde{\tau}$, *where* $\tilde{\tau}$ *is an activation threshold chosen close to* 0. *Let* $\boldsymbol{s_O} : [0, \tilde{\tau}] \to [0, \tilde{\beta}]$, *be a monotonically increasing function satisfying* $\boldsymbol{s_O}(0) = 0$.

The function $\boldsymbol{O^*_{\tilde{\tau}}} : [\mathbf{0},\mathbf{1}]^n \to [\mathbf{0},\mathbf{1}]$, *defined as:*

$$\boldsymbol{O^*_{\tilde{\tau}}}(\tilde{z}) = \begin{cases} \boldsymbol{s_O}(\boldsymbol{O}(\tilde{z})), & \textit{if } \boldsymbol{O}(\tilde{z}) \leq \tilde{\tau}, \\ \boldsymbol{O}(\tilde{z}), & \textit{if } \boldsymbol{O}(\tilde{z}) > \tilde{\tau}, \end{cases}$$

is termed as an activation-based overlap operator.

Let $\mathbf{O} : [\mathbf{0},\mathbf{1}]^n \to [\mathbf{0},\mathbf{1}]$ be an overlap operator. We can consider the activation function as the constant function $\boldsymbol{s_O}(t) = 0\ \forall\ t \in [0, \tilde{\tau}]$, then $\boldsymbol{O^*_{\tilde{\tau}}}$ reduces to the following form:

$$\boldsymbol{O^*_{\tilde{\tau}}}(\tilde{z}) = \begin{cases} 0, & \text{if } \mathbf{O}(\tilde{z}) \leq \tilde{\tau}, \\ \mathbf{O}(\tilde{z}), & \text{if } \mathbf{O}(\tilde{z}) > \tilde{\tau}\ . \end{cases}$$

From the construction of the function $\boldsymbol{O^*_{\tilde{\tau}}}$, it is clear that the function is symmetric and non-decreasing but need not be continuous.

Example IV.1. *The following are examples of activation-based overlap functions.*

1) *On considering the activation function*

$$\boldsymbol{s}_1\boldsymbol{(t)} = 1 - \left[\tilde{\beta} + (1-\tilde{\beta})\left(\frac{(1-t)-\tilde{\tau}}{1-\tilde{\tau}}\right)^{\alpha}\right], t \in [0, 1-\tilde{\tau}].$$

and the overlap operator $\boldsymbol{O}(\tilde{z})$, *defined as* $\boldsymbol{O}(\tilde{z}) = \prod\limits_i(z_i)$, *the following activation-based overlap operator is obtained:*

$$\boldsymbol{O^*_{\tilde{\tau}}}(\tilde{z}) = \begin{cases} \prod\limits_i(z_i), & \boldsymbol{O}(\tilde{z}) > \tilde{\tau} \\ \boldsymbol{P^*}(\tilde{z}) & \boldsymbol{O}(\tilde{z}) \leq \tilde{\tau}. \end{cases}$$

where $\boldsymbol{P^*}(\tilde{z}) = 1 - \left[\tilde{\beta} + (1-\tilde{\beta})\left(\frac{(1-\prod_i(z_i))-\tilde{\tau}}{1-\tilde{\tau}}\right)^{\alpha}\right]$, $\forall \tilde{z}$

2) *On considering the activation function*

$$\boldsymbol{s}_2(t) = 1 - \left[\tilde{\beta} + (1-\tilde{\beta})\,\frac{e^{\lambda((1-t)-\tilde{\tau})}-1}{e^{\lambda(1-\tilde{\tau})}-1}\right], t \in [0, 1-\tilde{\tau}], \lambda > 0,$$

and the overlap operator $\boldsymbol{O}(\tilde{z})$, *defined as* $\boldsymbol{O}(\tilde{z}) = \sqrt{\prod\limits_i(z_i)}$, *the following activation-based overlap operator is obtained*

$$\boldsymbol{O^*_{\tilde{\tau}}}(\tilde{z}) = \begin{cases} \sqrt{\prod\limits_i(z_i)}, & \boldsymbol{O}(\tilde{z}) > \tilde{\tau} \\ \boldsymbol{M^*}(\tilde{z}), & \boldsymbol{O}(\tilde{z}) \leq \tilde{\tau}. \end{cases}$$

where $\boldsymbol{M^*}(\tilde{z}) = 1 - \left[\tilde{\beta} + (1-\tilde{\beta})\,\frac{e^{\lambda\left((1-\sqrt{\prod_i(z_i)})-\tilde{\tau}\right)}-1}{e^{\lambda(1-\tilde{\tau})}-1}\right]$.

3) *On considering the activation function*

$$\boldsymbol{s}_3(t) = 0, \quad t \in [0, 1-\tilde{\tau}],$$

and the overlap operator $\boldsymbol{O}(\tilde{z})$, *defined as* $\boldsymbol{O}(\tilde{z}) = 1 - \frac{\max_i(1-z_i)}{\max_i(1-z_i)+\sqrt{\prod_i(z_i)}}$, *the following activation-based overlap operator is obtained:*

$$\boldsymbol{O^*_{\tilde{\tau}}}(\tilde{z}) = \begin{cases} 0, & \boldsymbol{O}(\tilde{z}) \leq \tilde{\tau}, \\ 1 - \frac{\max_i(1-z_i)}{\max_i(1-z_i)+\sqrt{\prod_i(z_i)}} & \boldsymbol{O}(\tilde{z}) > \tilde{\tau} \end{cases}$$

Theorem IV.1 (Duality Theorem)**.** *Let* $\boldsymbol{G^*_{\tilde{\tau}}} : [\mathbf{0},\mathbf{1}]^n \to [\mathbf{0},\mathbf{1}]$ *be an activation-based grouping operator constructed from a grouping operator* $\boldsymbol{G}$, *an activation threshold* $\tilde{\tau}$, *and an activation function* $\boldsymbol{s_G} : [\tilde{\tau}, 1] \to [\tilde{\beta}, 1]$, *where* $\tilde{\tau} \leq \tilde{\beta} \leq 1$, $\boldsymbol{s_G}(1) = 1$. *Define the function* $\boldsymbol{O^*_{\tilde{\tau}}} : [\mathbf{0},\mathbf{1}]^n \to [\mathbf{0},\mathbf{1}]$ *by* $\boldsymbol{O^*_{\tilde{\tau}}}(\tilde{z}) = 1 - \boldsymbol{G^*_{\tilde{\tau}}}(\tilde{1}-\tilde{z})$. *Then,* $\boldsymbol{O^*_{\tilde{\tau}}}$ *is an activation-based overlap operator, based on the activation function* $\boldsymbol{s_O} : [0, \tilde{\boldsymbol{\tau}}_O] \to [0, 1-\tilde{\beta}]$, *given by,* $\boldsymbol{s_O}(t) = 1 - \boldsymbol{s_G}(1-t)$ *and* $\tilde{\boldsymbol{\tau}}_O = 1 - \tilde{\tau}$.

Proof. We know that $\boldsymbol{s_G}$ is a monotonically increasing real valued function and moreover, $\boldsymbol{s_G}(1) = 1$. So, $\boldsymbol{s_O} : [0, \tilde{\tau}_O] \to [0, 1 - \tilde{\beta}]$, given by, $\boldsymbol{s_O}(t) = 1 - \boldsymbol{s_G}(1 - t)$, is also monotonically increasing and, $\boldsymbol{s_O}(0) = 0$. Now, using the definition of $\boldsymbol{G^*_{\tilde{\tau}}}$, we can conclude that the function $\boldsymbol{O^*_{\tilde{\tau}}}$ is an activation-based overlap operator with activation threshold $\tilde{\boldsymbol{\tau}}_O = 1 - \tilde{\tau}$, and activation function $\boldsymbol{s_O}$.

□

V. Conclusion

The main intention behind the proposed study is to develop a theoretical framework for an activation-based grouping and overlap operators. These operators are governed by a threshold parameter $\tilde{\tau}$, which determines the decision point from where the activation needs to be initiated. The value of this threshold depends on the underlying decision system which is under consideration. The construction methodology of the activation based grouping operators has been well established, and their fundamental properties have been extensively studied. The proposed operators are formulated in such a way that the output response are activated through suitable activation functions near the boundary region, while preserving the original behavior of the grouping or overlap operator in the non-activated regions. The proposed operators are sensitive near the threshold value, and thus the activation-based grouping and overlap operators exhibit strong potential for practical applications, particularly in decision-making, reliability analysis, information fusion, etc.

In future studies, the theory of activation-based grouping and overlap operators can be extended to ensemble learning and classification problems. Furtermore, such class of activation-based grouping operators can be utilized in feature extraction [22], pattern recognition [23], etc. Despite the promising theoretical results established in this work, certain limitations remain for such class of operators. In particular, the selection of an appropriate threshold value often requires empirical tuning due to the heterogeneous nature of real-world data, which is largely dependent on the necessities of the decision-system. Consequently, further research is needed to develop efficient and adaptive parameter learning strategies which are data-dependent to enable scalable and practical large-scale implementations [24].

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