Solving Fractional Diffusion Equation with Hilfer Derivative Operator using the Fuzzy Transform

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***Abstract*—In recent decades, fractional diffusion equations have found numerous applications in various scientific fields, particularly in industry, medicine, and finance, and especially in computer science, specifically image processing. This paper focuses on the fractional diffusion model and proposes a numerical method known as Fuzzy transform, to approximate its solution. The stability and convergence of the method are rigorously analyzed. Numerical results show that the method achieves a good balance between computational efficiency and accuracy. Thus, confirming its suitability for hybrid computing environments that combine artificial intelligence and numerical methods.**

***Index Terms*—fractional, diffusion, image processing, Fuzzy transform.**

I. Introduction

Fractional diffusion equations (FDEs) are generalizations of classical diffusion models in which integer-order derivatives are replaced by fractional-order derivatives (of order $\alpha < 2$). These equations are used to describe anomalous diffusion, in which particle propagation deviates from a Gaussian distribution. They are also used in physics, biology, and finance to model nonlocal transport, heterogeneous structures, and processes with memory effects, such as long-range pollutant flow and anomalous heat transfer.

The fractional diffusion equations have various applications in many scientific fields: image processing [1]–[3], industrial [4], financial [5], [6], medical and biological [7], etc.

According to [8], [9], the fractional derivative $D_{a+}^{\mu,\nu}$ of order $0 < \mu < 1$ and type $0 \leq \nu \leq 1$ with respect to x is defined as follows:

$$(D_{a+}^{\mu,\nu} f)(x) = (I_{a+}^{\nu(1-\mu)} \frac{d}{dx}(I_{a+}^{(1-\nu)(1-\mu)} f))(x), \qquad (1)$$

provided that the right-hand side exists. This generalized definition gives the classical Riemann-Liouville (R-L) fractional derivative operator if $\nu = 0$. For $\nu = 1$, this gives the fractional derivative operator introduced by Liouville, but nowadays often named after Caputo. Several authors called $D_{a+}^{\mu,\nu}$ *the Hilfer fractional derivative operator* or the composite fractional derivative operator. If μ or ν ia a non-integer number then the fractional derivative of function f is non-local, i.e. at any point $x = c$ it depends on all values of f, even those far from c.

Let $AC[a, b]$ be the space of real-valued functions $f(x)$ which are absolutely continuous on $[a, b]$. If $\phi \in AC[a, b]$, $(0 < \mu < 1)$ then using [8], [9], we can implicitly express $(D_{a+}^{\mu,\nu} f)(x)$ as follows:

$$(I_{a+}^{\mu} D_{a+}^{\mu} \phi)(x) = \phi(x) - [\lim_{x\to a+} (I_{a+}^{1-\mu} \phi)(x)] \frac{(x-a)^{\mu-1}}{\Gamma(\mu)}. \qquad (2)$$

This transformation will help us further derive the integral form of the diffusion equation, which is the main subject of our analysis.

II. Some properties of the Hilfer fractional diffusion equations

In this section, we provide an overview of the *Hilfer* fractional diffusion equations (the general form) and discuss some properties. We consider a fractional diffusion equation

$$D_{0+}^{\mu,\nu} u(x,t) = K \frac{\partial^2}{\partial x^2} u(x,t), \quad t > 0, x \in [0,1], \qquad (3)$$

under the initial condition $\lim_{t\to 0+} I_{0+}^{(1-\mu)(1-\nu)} u(x,t) = g(x)$, where K is a diffusion constant. First, we calculate the composition of the R-L fractional integral I_{0+}^{μ} and $D_{0+}^{\mu,\nu}$, using a semi-group property $I_{0+}^{\alpha} I_{0+}^{\beta} = I_{0+}^{\alpha+\beta}$, and a representation (2). Then, for any function $u(x,t)$ that satisfies the form (3), we obtain the following:

$$\begin{aligned}
(I_{0+}^{\mu} D_{0+}^{\mu,\nu} u)(x,t) &= (I_{0+}^{\mu} I_{0+}^{\nu(1-\mu)} D_{0+}^{\mu+\nu-\mu\nu} u)(x,t) \\
&= (I_{0+}^{\mu+\nu(1-\mu)} D_{0+}^{\mu+\nu-\mu\nu} u)(x,t) \\
&= u(x,t) - \frac{t^{-(1-\mu)(1-\nu)}}{\Gamma(-(1-\mu)(1-\nu)+1)} \lim_{t\to 0+} I_{0+}^{(1-\mu)(1-\nu)} u(x,t).
\end{aligned}$$

Applying operator I_{0+}^{μ} to both sides of (3) and using equation (2), we obtain the Volterra integro-differential equation:

$$\begin{aligned}
u(x,t) =& \frac{t^{-(1-\mu)(1-\nu)}}{\Gamma(-(1-\mu)(1-\nu)+1)} g(x) \\
&+ \frac{K}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} \frac{\partial^2}{\partial x^2} u(x,\tau) d\tau. \qquad (4)
\end{aligned}$$

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In the next section, we focus on a more specific class of fractional diffusion equations, namely those involving the Caputo fractional derivative.

III. The Caputo Fractional Diffusion Equation

In this section, we focus on equation (3), where we set $\nu = 1$ and restrict the time variable t to a finite interval $[0,1]$. This means that we add boundary conditions and replace the Hilfer derivative operator with the Caputo derivative [1]

$$ {}_C D^{\mu}_{0+} u(x,t) = I^{1-\mu}_{0^+} \frac{\partial u}{\partial t}(x,t). $$

After this, we consider equation (3) as a diffusion equation with Dirichlet boundary conditions:

$$\begin{cases} {}_C D^{\mu}_{0+} u(x,t) = K \frac{\partial^2}{\partial x^2} u(x,t), & t \in (0,1], x \in (0,1) \\ u(x,0) = g(x), & x \in [0,1], \\ u(0,t) = q_1(t), & t \in [0,1], \\ u(1,t) = q_2(t), & t \in [0,1], \end{cases} \tag{5}$$

where $K > 0$, $g \in C^4[0,1]$ and $q_1, q_2 \in C^1[0,1]$ so that

$$\begin{cases} g(0) = q_1(0), \\ g(1) = q_2(0). \end{cases}$$

It is known [14] that in the proposed formulation, there is a unique solution to the problem (5). This property is important for substantiating the fact that the numerical method proposed below leads to an exact solution. The latter is assumed to be sufficiently smooth.

Our main technical trick is to replace the differential equation in (5) with an integro-differential system containing Volterra and Fredholm integral operators. We first apply the operator I^{μ}_{0+} to both sides of the fractional Caputo diffusion equation in (5), similar to how we applied the same operator to (3) and obtained (4). After that we obtain the following Volterra integro-differential system where $x \in (0,1], t \in [0,1]$:

$$\begin{cases} u(x,t) = g(x) + \frac{K}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} \frac{\partial^2}{\partial x^2} u(x,\tau) d\tau, \\ u(0,t) = q_1(t), \\ u(1,t) = q_2(t). \end{cases} \tag{6}$$

From now on, our methodology will focus on transforming (6) into a form that includes only integral operators. The main idea consists in introducing a new unknown function $y(x,t) = \frac{\partial^2}{\partial x^2} u(x,t)$, where $x \in (0,1]$ and $t \in [0,1]$, and rewriting the problem (6) assuming that the function $y(x,\cdot)$ is continuous in a neiborhood of 0, i.e. in $(0,1]$. For a fixed $t \in [0,1]$, using the Taylor expansion formula for $x \in (0,1]$ with the remainder in integral form, we obtain the following representation:

$$ u(x,t) = q_1(t) + x u_x(0,t) + \int_0^x (x-\xi) y(\xi,t) d\xi, \tag{7}$$

[1] In this text, the Caputo derivative is used to simplify the description of the proposed method. As we saw in the previous section, the result of applying the fractional integral I^{μ}_{0+} to the Hilfer or Caputo derivative preserves the leading term and differs in the term related to the initial condition. This has virtually no effect on the proposed method, as can be verified in the final section using numerical tests.

which for $x = 1$ changes to:

$$ u(1,t) = q_2(t) = q_1(t) + u_x(0,t) + \int_0^1 (1-\xi) y(\xi,t) d\xi, $$

and then, by the boundary condition $u(1,t) = q_2(t)$, on

$$ u_x(0,t) = q_2(t) - q_1(t) - \int_0^1 (1-\xi) y(\xi,t) d\xi. \tag{8}$$

Substituting (8) for $u_x(0,t)$ in (7), we get

$$\begin{aligned} u(x,t) =& (1-x) q_1(t) + x q_2(t) + \int_0^x (x-\xi) y(\xi,t) d\xi \\ & - x \int_0^1 (1-\xi) y(\xi,t) d\xi. \end{aligned} \tag{9}$$

Now we replace the function u in (6) with its new representation from (9) and rewrite (6) in terms of the new unknown function y:

$$\begin{aligned} \int_0^x (x-\xi) y(\xi,t) d\xi - \frac{K}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} y(x,\tau) d\tau \\ - x \int_0^1 (1-\xi) y(\xi,t) d\xi \\ = g(x) - (1-x) q_1(t) - x q_2(t). \end{aligned} \tag{10}$$

Finally, we replace y with $\overline{y}$, where

$$ \overline{y}(x,t) = y(x,t) - g''(x), \tag{11}$$

so that the new function has zero initial condition $\lim_{t \to 0^+} \overline{y}(x,t) = 0$, which follows from (5).

To write the new form of the system (6), we denote

- $I_1(\overline{y})(x,t) = \int_0^x (x-\xi) \overline{y}(\xi,t) d\xi$,
- $I_2(\overline{y})(x,t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} \overline{y}(x,\tau) d\tau$,
- $I_3(\overline{y})(x,t) = x \int_0^1 (1-\xi) \overline{y}(\xi,t) d\xi$.

Moreover, we denote the function $\rho(x,t)$ as follows:

$$\begin{aligned} \rho(x,t) =& g(x) - (1-x) q_1(t) - x q_2(t) + \frac{K g''(x)}{\Gamma(\mu+1)} t^{\mu} \\ & + \left(x \int_0^1 (1-\xi) g''(\xi) d\xi - \int_0^x (x-\xi) g''(\xi) d\xi \right). \end{aligned}$$

Then, the new form of the system (6) consists of only one equation

$$ I_1(\overline{y})(x,t) - I_3(\overline{y})(x,t) - K I_2(\overline{y})(x,t) = \rho(x,t), \tag{12}$$

where the unknown function $\overline{y}$ must be found for $x \in [0,1], t \in [0,1]$.

After obtaining the solution $\overline{y}$ to the problem (12), we first evaluate the intermediate function y using (11) and then, using (9), obtain the desired solution u to the problem (6) as follows:

$$\begin{aligned} u(x,t) =& (1-x) q_1(t) + x q_2(t) + \int_0^x (x-\xi) y(\xi,t) d\xi \\ & - x \int_0^1 (1-\xi) y(\xi,t) d\xi. \end{aligned}$$

Thus, the proposed method for solving the problem (5) consists of finding a solution $\overline{y}$ of the problem (12), and then applying the equalities (11) and (9) to find the desired solution u of the problem (5).

Since in most cases the exact solution to the considered problem (5) and the related to it problem (12) cannot be expressed analytically, numerical methods are needed to find approximate solutions. In the next section, we present a powerful and flexible method known as fuzzy (F-)transform for finding approximate solutions to the problem (12) .

IV. Proposed method (Fuzzy transform)

The F-transform provides a stable and flexible approximation framework that does not require high smoothness of the underlying function, making it particularly suitable for problems with weak singularities and memory effects, such as fractional differential and integral equations. The theory of fuzzy transforms was originally proposed by I. Perfilieva in [10] and then further generalized in [11]. This theory provides a good example of how fuzzy modeling can be applied to solving classical problems using efficient numerical methods. Similar examples of applying fuzzy methodology to conventional problems can be found, for example, in [15], [16].

For the purpose of solving integral equations with a weakly singular kernel we introduce a *graded fuzzy partition*:

Definition 1: [12]

- Let $0 < \varepsilon < 1$, $n = \left\lceil \frac{1}{\varepsilon} \right\rceil$, $\gamma \geq 1$.
- Choose nodes $t_i = \left(\frac{i}{n}\right)^\gamma$, $i = 1, \cdots, n$.
- Fuzzy sets $A_0, A_1 \cdots, A_n : [0,1] \to [0,1]$, identified with their continuous membership functions, constitute a fuzzy partition of $[0,1]$ with nodes $0 = t_0 < t_1 < \cdots < t_n = 1$, if they fulfill the following conditions:
    1) (locality) For all $k = 1, \cdots, n-1$, $A_k(t) = 0$ if $t \notin (t_{k-1}, t_{k+1})$; $A_0(t) = 0$ if $t \notin [0, t_1)$; $A_n(t) = 0$ if $t \notin (t_n, 1]$;
    2) (positiveness) For all $k = 1, \cdots, n-1$, $A_k(t) > 0$ if $t \in (t_{k-1}, t_{k+1})$, $A_0(t) > 0$ if $t \in [0, t_1)$; $A_n(t) > 0$ if $t \in (t_{n-1}, t_n]$;
    3) (normality) For all $k = 0, \cdots, n$, $A_k(t_k) = 1$.

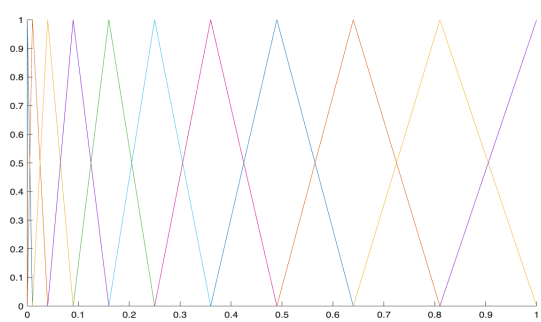

Fig. 1. The graded partition with $\gamma = 2$.

Remark: If $\gamma = 1$, we have the h-uniform fuzzy partition with $h = \frac{1}{n}$.

Definition 2: [11] Let $n \geq 2$, and $A_0, \ldots, A_n$ be a fuzzy partition of $[0,1]$ with nodes $t_0, t_1, \ldots, t_n$. Let function $f \in L^2[0,1]$. Then,

1) the (direct) F^0-transform of f is a vector of real numbers $(F_0^0, \ldots, F_n^0)$ such that

$$F_k^0 = \frac{\int_{x_{k-1}}^{x_{k+1}} f(t) A_k(t) dt}{\int_{x_{k-1}}^{x_{k+1}} A_k(t) dt},$$

where we set $t_{-1} = t_0 = 0$ and $t_{n+1} = t_n = 1$;

2) the inverse F^0-transform of f is a function $\hat{f}_F^0 \in C[0,1]$ represented as

$$\hat{f}_F^0(t) = \sum_{k=0}^{n} F_k^0 A_k(t),$$

where $(F_0^0, \ldots, F_n^0)$ is the direct F^0-transform of f.

In the next section, we apply the F-transform to find an approximate solution of (12).

V. Approximate solution. Algorithm

The proposed algorithm is hybrid, as it combines theoretical pre-processing and, based on it, a numerical method. Before applying the F-transform to both sides of the equation (12), we choose fuzzy partitions of the intervals $[0,1]$ on both the t and x axes as follows:

- For a positive integer $m \geq 2$, let $0 = x_0 < x_1 < \ldots < x_m = 1$ be nodes and $B_0(x), \ldots, B_m(x)$ be the $\frac{1}{m}$-uniform fuzzy partition of $[0,1]$ with the triangular shapes.
- For a positive integer $n \geq 2$ and $\gamma > 1$, let $t_i = \left(\frac{i}{n}\right)^\gamma$, $i = 0, \cdots, n$ be nodes of a graded fuzzy partition $[0,1]$ with $A_0(t), \cdots, A_n(t)$ as the corresponding triangular basic functions (see [12] for details).

Let us denote $\mathbf{A}(t) = \begin{pmatrix} A_0(t) & A_1(t) & \cdots & A_n(t) \end{pmatrix}^T$ and $\mathbf{B}(x) = \begin{pmatrix} B_0(x) & B_1(x) & \cdots & B_m(x) \end{pmatrix}^T$. We will proceed in 3 steps:

1) Let us approximate $I_2(x,t)$ first. Fix a constant l ($l = 0, \ldots, m$), denote $\overline{y}_l(t) = \overline{y}(x_l, t)$. Then, based on [12], we obtain the following result

$$\begin{pmatrix} I_2(x_l, t_0) \\ I_2(x_l, t_1) \\ \vdots \\ I_2(x_l, t_n) \end{pmatrix} = \begin{pmatrix} \mathcal{I}_{0+}^{\mu}(y_l)(t_0) \\ \mathcal{I}_{0+}^{\mu}(y_l)(t_1) \\ \vdots \\ \mathcal{I}_{0+}^{\mu}(y_l)(t_n) \end{pmatrix} \approx L^\mu \begin{pmatrix} Y_{l,0}^0 \\ Y_{l,1}^0 \\ \vdots \\ Y_{l,n}^0 \end{pmatrix},$$

where L^μ is defined in [12]. Then, denote $\mathbf{Y}$ the $n \times m$ matrix of F^0-transform components of $\overline{y}(x,t)$.

$$\mathbf{I}_2 = \begin{pmatrix} I_2(x_0,t_0) & I_2(x_1,t_0) & \cdots & I_2(x_m,t_0) \\ I_2(x_0,t_1) & I_2(x_1,t_1) & \cdots & I_2(x_m,t_1) \\ \vdots & \vdots & & \vdots \\ I_2(x_0,t_n) & I_2(x_1,t_n) & \cdots & I_2(x_m,t_n) \end{pmatrix}$$

$$\approx L^\mu \begin{pmatrix} Y_{0,0}^0 & Y_{1,0}^0 & \cdots & Y_{m,0}^0 \\ Y_{0,1}^0 & Y_{1,1}^0 & \cdots & Y_{m,1}^0 \\ \vdots & \vdots & & \vdots \\ Y_{0,n}^0 & Y_{1,n}^0 & \cdots & Y_{m,n}^0 \end{pmatrix} = L^\mu \mathbf{Y}. \quad (13)$$

2) Then, we approximate $I_1(x,t)$. Let us denote $\kappa(x,\xi) = (x-\xi)$ and fix a constant k ($k = 0,\cdots,n$) and denote $\overline{y}^k(x) = \overline{y}(x,t_k)$. Then, we have

$$I_1(x,t_k) = \mathcal{V}_\kappa \overline{y}^k(x) = \int_0^t \kappa(x,\xi)\overline{y}^k(\xi)d\xi.$$

Then,

$$\begin{aligned} I_1(x_l,t_k) &= \mathcal{V}_\kappa \overline{y}^k(x_l) \\ &\approx \begin{pmatrix} Y^0_{0,k} & Y^0_{1,k} & \cdots & Y^0_{m,k} \end{pmatrix} (\mathcal{K}^T \odot P)\mathbf{B}(x_l). \end{aligned}$$

Then,

$$\mathbf{I}_1 = \begin{pmatrix} I_1(x_0,t_0) & I_1(x_1,t_0) & \cdots & I_1(x_m,t_0) \\ I_1(x_0,t_1) & I_1(x_1,t_1) & \cdots & I_1(x_m,t_1) \\ \vdots & \vdots & & \vdots \\ I_1(x_0,t_n) & I_1(x_1,t_n) & \cdots & I_1(x_m,t_n) \end{pmatrix}$$

$$\approx \begin{pmatrix} Y^0_{0,0} & Y^0_{1,0} & \cdots & Y^0_{m,0} \\ Y^0_{0,1} & Y^0_{1,1} & \cdots & Y^0_{m,1} \\ \vdots & \vdots & & \vdots \\ Y^0_{0,n} & Y^0_{1,n} & \cdots & Y^0_{m,n} \end{pmatrix} (\mathcal{K}^T \odot P) \tag{14}$$

$$= \mathbf{Y}(\mathcal{K}^T \odot P), \tag{15}$$

where $\mathcal{K}$ and P are defined in [13] and $\odot$ is Hadamard product. Let us denote $\mathbf{P} = \begin{pmatrix} P^0_{0,0} & P^0_{1,0} & \cdots & P^0_{m,0} \\ P^0_{0,1} & P^0_{1,1} & \cdots & P^0_{m,1} \\ \vdots & \vdots & & \vdots \\ P^0_{0,n} & P^0_{1,n} & \cdots & P^0_{m,n} \end{pmatrix}$ be matrix of Fuzzy transform components of $\rho(x,t)$.

3) The last step is approximate $I_3(x,t)$. Fix any $l = 0,\cdots,m$ and $k = 0,\cdots,n$, and denote $\varphi_k = \int_0^1 (1-\xi)\overline{y}(\xi,t_k)d\xi$ we have

$$I_3(x_l,t_k) = \left(\int_0^1 (1-\xi)\overline{y}(\xi,t_k)d\xi\right) x_l = \varphi_k x_l.$$

Let us denote $\mathbf{x} = \begin{pmatrix} x_0 & x_1 & \cdots & x_m \end{pmatrix}$. Then, we have

$$\begin{pmatrix} I_3(x_0,t_k) & I_3(x_1,t_k) & \cdots & I_3(x_m,t_k) \end{pmatrix} \tag{16}$$

$$= \varphi_k \begin{pmatrix} x_0 & x_1 & \cdots & x_m \end{pmatrix} = \varphi_k \mathbf{x}. \tag{17}$$

In addition, denote $\Omega = \begin{pmatrix} \omega_0 & \omega_1 & \cdots & \omega_m \end{pmatrix}$ where

$$\begin{aligned} \omega_l &= \int_0^1 (1-\xi)B_l(\xi)d\xi \\ &= \begin{cases} \frac{h}{2} - \frac{h^2}{6}, & l = 0, \\ h(1-lh), & l = 1,\cdots,m-1, \\ \frac{h^2}{6}, & l = m. \end{cases} \end{aligned}$$

Then, the number φ_k could be approximate as follows by Fuzzy-transform

$$\varphi_k = \int_0^1 (1-\xi)\overline{y}^k(\xi)d\xi \approx \begin{pmatrix} Y^0_{0,k} & Y^0_{1,k} \cdots & Y^0_{m,k} \end{pmatrix} \Omega^T.$$

Then, vector (16) could be approximate as

$$\begin{aligned} &\begin{pmatrix} I_3(x_0,t_k) & I_3(x_1,t_k) & \cdots & I_3(x_m,t_k) \end{pmatrix} \\ &\approx \begin{pmatrix} Y^0_{0,k} & Y^0_{1,k} \cdots & Y^0_{m,k} \end{pmatrix} \Omega^T \mathbf{x}. \end{aligned}$$

Then,

$$\mathbf{I}_3 = \begin{pmatrix} I_3(x_0,t_0) & I_3(x_1,t_0) & \cdots & I_3(x_m,t_0) \\ I_3(x_0,t_1) & I_3(x_1,t_1) & \cdots & I_3(x_m,t_1) \\ \vdots & \vdots & & \vdots \\ I_3(x_0,t_n) & I_3(x_1,t_n) & \cdots & I_3(x_m,t_n) \end{pmatrix}$$

$$\approx \begin{pmatrix} Y^0_{0,0} & Y^0_{1,0} & \cdots & Y^0_{m,0} \\ Y^0_{0,1} & Y^0_{1,1} & \cdots & Y^0_{m,1} \\ \vdots & \vdots & & \vdots \\ Y^0_{0,n} & Y^0_{1,n} & \cdots & Y^0_{m,n} \end{pmatrix} \Omega^T \mathbf{x} = \mathbf{Y}\Omega^T \mathbf{x}.$$

Then, we replace $\mathbf{I}_1$ as (14) and $\mathbf{I}_2$ as (13) with the corresponding factors in (12). Then, equation (12) could be approximated as

$$\mathbf{Y}\left((\mathcal{K}^T \odot P) - \Omega^T \mathbf{x}\right) - KL^\mu \mathbf{Y} = \mathbf{P}. \tag{18}$$

Then, we use the vectorization operator vec. Let $\mathcal{H} = \left((\mathcal{K}^T \odot P)^T - \mathbf{x}^T \Omega\right) \otimes I_{n\times n} - I_{m\times m} \otimes KL^\mu$. Then,

$$vec(\mathbf{Y}) = \mathcal{H}^{-1} vec(\mathbf{P}).$$

Then, by the inverse F^0-transform, we obtain the approximation $\overline{y}_{ap}$ of $\overline{y}$.

$$\overline{y}_{ap}(x,t) = \mathbf{A}(t)\mathbf{Y}\mathbf{B}(x).$$

Finally, as we described at the end of Section III, after using (11) and (9), we find an approximate solution $u_a p$ to the diffusion equation in (5).

In the next section, we consider two examples illustrating the effectiveness of the proposed method, which is achieved by applying the F-transform to find approximate solutions.

VI. Numerical Tests

Example 1: Let us consider the problem (5) where $\mu = 1/2$:

$$\begin{cases} g(x) = x^4, \\ q_1(t) = \frac{(\Gamma(\mu+1))^2}{24\Gamma(2\mu+1)} t^{2\mu}, \\ q_2(t) = 1 + \frac{t^\mu}{2} + \frac{(\Gamma(\mu+1))^2}{24\Gamma(2\mu+1)} t^{2\mu}, \end{cases}$$

and the exact solution is

$$u(x,t) = x^4 + \frac{x^2 t^\mu}{2} + \frac{(\Gamma(\mu+1))^2}{24\Gamma(2\mu+1)} t^{2\mu}.$$

In accordance with the proposed methodology, we first transform the system under consideration into the equation (12) with a new unknown function $\overline{y}$, characterized in (11). The right-hand side of (12) is $\rho(x,t) = \frac{x^2 t^\mu}{2} - \frac{x t^\mu}{2} - t^{2\mu}$. After that we apply the F-transform and find the F^0-transform approximation $\mathbf{Y}$ of the auxiliary solution $\overline{y}$.

In more detail, for any number n, we set $m = n+1$ and form the partitions $x_0,\cdots,x_{n+1}$ and $t_0,\cdots,t_n$, as discussed in the proposed method. After applying the F^0-transform and finding an approximation of $\mathbf{Y}$, we stop further calculations, since the exact solution u (known for this example) can be used to calculate the exact solution $\overline{y}$ of the equation (12). In this example,

$$\overline{y}(x,t) = u_{xx}(x,t) - g''(x) = t^{2\mu}. \tag{19}$$

Therefore, the quality of the approximation will be assessed using the L^1 norm, which measures the average absolute difference between the approximate and exact solutions. To do this, we construct a matrix $\mathbf{Ex}$ whose elements are $\overline{y}(x_i, t_j)$, $i = 0, \cdots, n+1$, $j = 0 \cdots, n$, and compare it with $\mathbf{Y}$:

$$\|\overline{y}_{ap} - \overline{y}\|_{L^1} = \frac{\sum_{i,j} |\mathbf{Y}_{i,j} - \mathbf{Ex}_{i,j}|}{n(n+1)}. \tag{20}$$

Table I shows the errors between the exact values of $\overline{y}$ and their approximations obtained using our proposed method.

TABLE I
THE L^1 ERROR BETWEEN $\overline{y}$ AND ITS APPROXIMATION $\overline{y}_{ap}$ IN EX. 1

n	L^1 **Error**
2	$4.3548e-2$
4	$1.2953e-2$
5	$8.6385e-3$
10	$2.3861e-3$
20	$6.4529e-4$

The Figure 2 shows the errors between $\overline{y}$ and its approximation $\overline{y}_{ap}$ in the form of the error curves $|\overline{y}(x,1) - \overline{y}_{ap}(x,1)|$, taken on the interval $[0,1]$ and for n equal to $2, 4, 5, 10$ and 20, respectively. In this figure, we see a rapid decrease in the approximation errors as the number n of elements in the fuzzy partition increases. The following colors are used for this purpose:

- Blue line: $n = 2$,
- Red line: $n = 4$,
- Yellow line: $n = 5$,
- Magenta line: $n = 10$,
- Green line: $n = 20$.

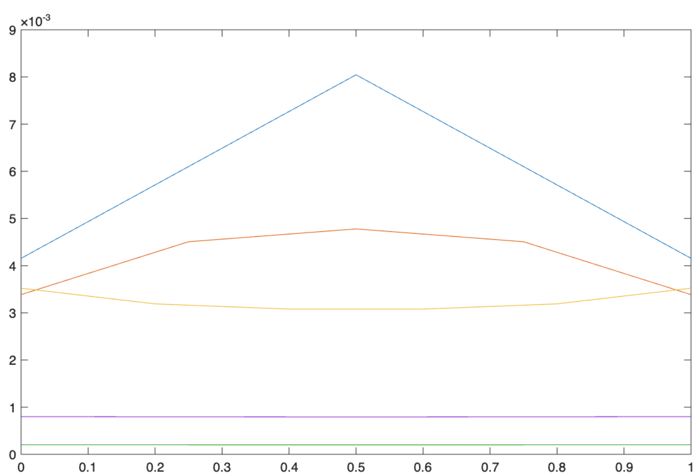

Fig. 2. The error curves $|\overline{y}(x,1) - \overline{y}_{ap}(x,1)|$ at the time $t = 1$ in Ex. 1.

Figure 3 shows 2D plots of the approximate and exact solutions of equation (12), which corresponds to the fractional diffusion equation in (5), considered in Example 1.

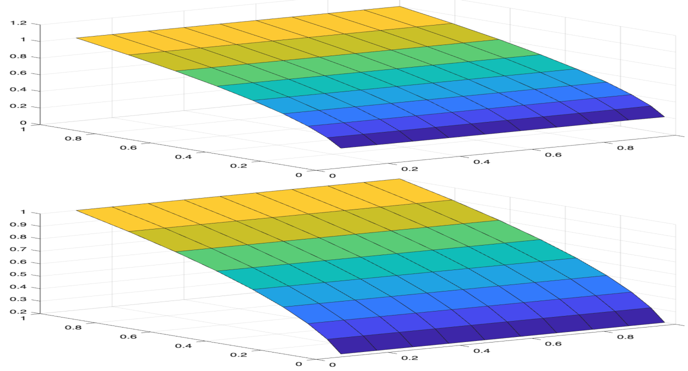

Fig. 3. The graphs of the approximate $\overline{y}_{ap}$ (top) and exact $\overline{y}$ (bottom) solutions of the equation (12) considered in Example 1.

Example 2: In this example, we consider the fractional diffusion equation (3) with the Hilfer derivative and the corresponding initial-boundary value problem:

$$\begin{cases} D^{\mu;\nu}_{0+} u(x,t) = K \frac{\partial^2}{\partial x^2} u(x,t) + f(x,t), & t \in [0,1], x \in [0,1], \\ \lim_{t \to 0+} I^{(1-\mu)(1-\nu)}_{0+} u(x,t) = g(x), & x \in [0,1], \\ u(0,t) = q_1(t), & t \in [0,1], \\ u(1,t) = q_2(t), & t \in [0,1]. \end{cases} \tag{21}$$

where $f(x,t) \in C^4\big([0,1], C^1(0,1]\big)$, $g \in C^4[0,1]$, q_1, $q_2 \in C^1[0,1]$ are given functions, u is an unknown function, and $K > 0$ is a positive number. We show that the proposed method is fully applicable to this case, justifying our claim in Section III.

The above equation will use the following parameters:

$$\begin{cases} K = \frac{1}{20}, \\ f(x,t) = \Gamma(\mu+1)(1+x^2) - \frac{t^\mu}{20}, \\ g(x) = 0, \\ q_1(t) = t^\mu, \\ q_2(t) = 2t^\mu. \end{cases} \tag{22}$$

which has the exact solution $u(x,t) = t^\mu(1+x^2)$. In the calculations we used the value $\mu = 0.5$. Similar to Example 1, we use the (20) norm to measure the errors of the exact and approximate auxiliary solutions and illustrate this in Table II.

TABLE II
THE L^1 ERROR BETWEEN $\overline{y}$ AND ITS APPROXIMATION $\overline{y}_{ap}$ IN EX. 2

n	L^1 **Error**
2	$6.9032e-2$
4	$1.5844e-2$
5	$9.7950e-3$
10	$2.2130e-3$
20	$5.2271e-4$

The Figure 4 shows the errors between $\overline{y}$ and its approximation $\overline{y}_{ap}$ in the form of the error curves $|\overline{y}(x,1) - \overline{y}_{ap}(x,1)|$, taken on the interval $[0,1]$ and for n equal to $2, 4, 5, 10$ and 20, respectively. In this figure, we see a rapid decrease in the approximation errors as the number n of elements in the fuzzy partition increases. The following colors are used for this purpose:

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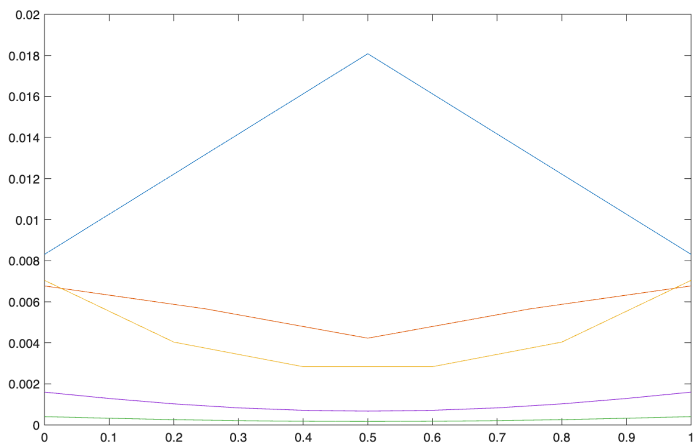

Fig. 4. The error curves $|\overline{y}(x,1) - \overline{y}_{ap}(x,1)|$ at the time $t = 1$ in Ex. 1.

VII. Conclusion

In this paper, we demonstrated that fuzzy modeling methods, and especially the F-transform method, can be successfully applied to solving classical problems, particularly those in which ordinary differential and integral calculus is generalized to its fractional form. Numerical methods are particularly in demand for problems of this type. We analyzed a class of diffusion equations with fractional partial derivatives and demonstrated how they can be efficiently solved numerically using the fuzzy transform methodology. Implementation of the proposed algorithm using neural networks, and particularly PINN, is a topic for future research.

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References

[1] J. Gao, J. Sun, S. Shi, "Fractional-order cross-diffusion system for multiplicative noise removal", Computers & Mathematics with Applications (164), (2024), 1-11, ISSN 0898-1221, https://doi.org/10.1016/j.camwa.2024.03.031.

[2] J. Bai, X. -C. Feng, "Fractional-Order Anisotropic Diffusion for Image Denoising," in IEEE Transactions on Image Processing (16), (2007), 2492-2502, doi: 10.1109/TIP.2007.904971

[3] W. Zhang, J. Li, Y. Yang, "Spatial fractional telegraph equation for image structure preserving denoising", Signal Processing (107), (2015), 368-377, ISSN 0165-1684, https://doi.org/10.1016/j.sigpro.2014.04.015.

[4] A. Toma, D. Mozyrska, O. Postavaru, M. Rebenciuc, S. M. Bibic, "Application of Fractional Calculus as an Interdisciplinary Modeling Framework", Fractal and Fractional (10), (2025) https://doi.org/10.3390/fractalfract9100663

[5] Z. Tomovski, J.L.A. Dubbeldam, J. Korbel, "Applications of Hilfer-Prabhakar Operator to Option Pricing Financial Model". Fract Calc Appl Anal (23), 9961012 (2020). https://doi.org/10.1515/fca-2020-0052

[6] F. Norouzi, G. M. Guerekata, "A study of ψ-Hilfer fractional differential system with application in financial crisis", Chaos, Solitons & Fractals (6), (2021),100056,ISSN 2590-0544, https://doi.org/10.1016/j.csfx.2021.100056.

[7] Shyamsunder, S. Bhatter, Kamlesh Jangid, S.D. Purohit, "Fractionalized mathematical models for drug diffusion", Chaos, Solitons & Fractals (165), (2022), 112810, ISSN 0960-0779, https://doi.org/10.1016/j.chaos.2022.112810.

[8] T. Sandev, Z. Tomovski, "Fractional Equations and Models"; Springer Nature: Durham, NC, United States, 2019; ISBN: 9783030296131.

[9] Z. Tomovski, D. Leskovski, S. Gerhold, " Generalized Mathieu Series". Springer Nature (2021). https://doi.org/10.1007/978-3-030-84817-0

[10] I.Perfilieva, "Fuzzy transforms: Theory and applications," Fuzzy Sets and Systems, vol. 157, 2006, 993-1023.

[11] I. Perfilieva, M. Dankova, B. Bede, "Towards a higher degree F-transform", Fuzzy Sets and Systems 180, 3-19 (2011). ISSN 0165-0114, https://doi.org/10.1016/j.fss.2010.11.002.

[12] I. Perfilieva, T.M.T. Pham, "On weakly singular Volterra equations in the direct sum of Hilbert spaces: Operational-matrix approach", submitted.

[13] I. Perfilieva, S. Ziari, R. Nuraei, T.M.T. Pham, "F-transform utility in the operational-matrix approach to the Volterra integral equation," FSS 475, 108764, (2024). ISSN 0165-0114.

[14] A. Kubica, M. Yamamoto, " Initial-boundary Value Problems for Fractional Diffusion Equations with Time-Dependent Coefficients". FCAA 21, 276311 (2018). https://doi.org/10.1515/fca-2018-0018

[15] C. Chen, C. Wagner, J. Garibaldi, "Gradient-based Fuzzy System Optimisation via Automatic Differentiation–FuzzyR as a Use Case", arXiv preprint arXiv:2403.12308, (2024). https://doi.org/10.48550/arXiv.2403.12308

[16] L. Ferranti, T. Runkler, "Automated Tuning of Fuzzy Systems to Solve Ordinary Differential Equations", in IEEE Int. Conf. Fuzzy Syst., FUZZ-IEEE, (2025). https://doi.org/10.1109/FUZZ62266.2025.11152176