

Prior-Driven Tensor RPCA: A New Regularization Approach for High-Dimensional Data Representation

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Abstract—Tensor Robust Principal Component Analysis (TRPCA) is an important technique in computer vision and machine learning, with numerous applications such as color image denoising and video background separation. Tensor nuclear norm minimization (TNNM) is a classical approach for TRPCA that has been shown to recover low-rank and sparse tensors under certain conditions. However, TNNM penalizes all rank components equally, ignoring their relative importance, which often leads to suboptimal recovery results. To overcome this limitation, we propose a novel truncated nonconvex fractional function to approximate the tensor multi-rank function. Benefiting from the truncation strategy and the nonconvex fractional structure, the proposed function preserves dominant singular values while adaptively penalizing less significant ones. Based on this function, we formulate a TRPCA model for separating low-rank structures and sparse noise from tensor data. An efficient algorithm based on the alternating direction method of multipliers (ADMM) is developed, and its computational complexity is analyzed. Extensive experiments on both synthetic and real-world datasets demonstrate that the proposed method outperforms several state-of-the-art approaches.

Index Terms—Tensor robust principal component analysis, Truncated nonconvex fraction function, Color image denoising, Video background subtraction

I. INTRODUCTION

Tensors, as a natural representation of multi-dimensional data, have been widely used in many applications, such as communication systems [1], machine learning [2]–[4], video background subtraction [5]–[7] and image/video denoising [8]–[12]. However, the collected tensor data is often contaminated by noise or outliers during acquisition and transmission, which severely affects data analysis and applications. To address this challenge, robust principal component analysis (RPCA) [13], [14] has been extensively studied, which aims to decompose the observed data into a low-rank component and a sparse component. Nevertheless, due to the inability to fully exploit the inherent multi-dimensional structure of real-world data, RPCA, which is limited by matrix representation, has naturally been extended to tensor RPCA (TRPCA). Generally, the TRPCA problem can be modeled as follows:

$$\min_{\mathcal{X}, \mathcal{E}} \text{rank}(\mathcal{X}) + \lambda \|\mathcal{E}\|_0 \quad s.t. \mathcal{D} = \mathcal{X} + \mathcal{E}, \quad (1)$$

where $\text{rank}(\cdot)$ denotes the tensor rank function, $\|\cdot\|_0$ denotes the ℓ_0 -norm, and λ is a positive weighting parameter. Unfortunately, directly solving (1) is computationally prohibitive, since both tensor rank minimization and ℓ_0 -norm minimization are NP-hard problems [15]. To address this problem, Lu *et al.* [7] utilized the tensor nuclear norm (TNN) and ℓ_1 -norm to convexly approximate the tensor rank function and promote sparse solutions, respectively, thus obtaining the following optimization problem:

$$\min_{\mathcal{X}, \mathcal{E}} \|\mathcal{X}\|_* + \lambda \|\mathcal{E}\|_1 \quad s.t. \mathcal{D} = \mathcal{X} + \mathcal{E}. \quad (2)$$

However, the TNN is only a rough approximation, which may lead to suboptimal solutions [16]. This is because, in practical applications, there are significant differences between the singular values of a tensor, with larger singular values containing the main structural information. The nuclear norm treats all singular values equally [17], leading to larger singular values being excessively penalized, thus introducing unavoidable bias.

To overcome this limitation, various improved or nonconvex relaxation methods have been developed to solve the low-rank tensor recovery problem [18]–[22]. For instance, Luan *et al.* [18] proposed a TRPCA model based on truncated nuclear norm and truncated Frobenius norm, which improves the accuracy of low-rank recovery by minimizing min singular values. However, the hybrid truncation strategy still treats the rank components uniformly, which may fail to fully exploit their relative importance, thus limiting the separation accuracy. Zheng *et al.* [20] proposed a TRPCA model based on the weighted sum of the tensor nuclear norm (WSTNN), which improves the accuracy of low-rank and sparse decomposition by assigning different weights to tensor nuclear norms along different modes. Moreover, Gao *et al.* [22] developed an enhanced TRPCA (ETRPCA) model, which achieves better preservation of important structural information in tensor data by assigning different weights to singular values. Nevertheless, the performance of the WSTNN and ETRPCA methods is highly dependent on the choice of weight parameters, which limits the applicability of the models in practical applications. In summary, the above methods either ignore the prior rank information, which leads to inaccurate separation

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of low-rank and sparse components, or lack robustness in practical applications, thereby resulting in unstable separation performance.

Motivated by the above limitations, we propose a novel truncated nonconvex fractional function to approximate the tensor multi-rank function. This function has the following three major advantages: 1) Low-rank prior preservation. By introducing the truncation strategy, the proposed function preserves the dominant singular values, which generally characterize the main structural information of a tensor, thereby alleviating the over-shrinkage issue observed in convex surrogates. 2) Adaptive penalization mechanism. Owing to its nonconvex fractional structure, the proposed penalty function can adaptively penalize different singular values, thereby effectively exploiting the prior information of different tensor rank components. 3) Computational efficiency. The proximal operator associated with the proposed penalty admits a closed-form solution, which enables each subproblem to be solved efficiently. Building upon this function, we formulate a low-rank minimization model for robust tensor principal component analysis, which can more accurately separate the low-rank and sparse components. Subsequently, an efficient algorithm based on the ADMM framework is developed to solve the proposed model. Finally, experimental results on both synthetic and real-world datasets demonstrate that the proposed method outperforms several state-of-the-art competing approaches. The source code is publicly available at https://github.com/wangzhi-swu/TRPCA_TNFF.

II. TRUNCATED NONCONVEX FRACTION FUNCTION MINIMIZATION

As is well known, TNN have been widely used in various low-rank tensor recovery tasks, such as image processing [23], [24], low-rank tensor estimation [25], subspace clustering [26], [27]. These tasks are generally formulated as solving the following TNNM problem:

$$\arg \min_{\mathcal{X}} \|\mathcal{X}\|_* + \frac{1}{2} \|\mathcal{X} - \mathcal{A}\|_F^2. \quad (3)$$

However, TNN treat all singular values of tensor equally. This is unreasonable because, in real-world applications, there are significant differences among non-zero singular values of a tensor, which is generally referred to as prior information. This means that the model (3) ignores this prior information, leading to biased estimation.

To overcome this limitation, we propose the following nonconvex fractional function to approximate the tensor rank norm.

Definition 1. For $\mathcal{X} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$, $h = \min(n_1, n_2)$, the nonconvex fractional function $\mathcal{P}_{a,r}(\mathcal{X})$ of $\mathcal{X}$ is defined as

$$\mathcal{P}_{a,r}(\mathcal{X}) = \frac{1}{n_3} \sum_{k=1}^{n_3} P_{a,r}(\bar{X}^{(k)}) = \frac{1}{n_3} \sum_{k=1}^{n_3} \sum_{i=r+1}^h \rho_a(\sigma_i(\bar{X}^{(k)})), \quad (4)$$

where $\rho_a(x) = \frac{a|x|}{1+a|x|}$, $a > 0$, and $\sigma_i(\bar{X}^{(k)})$ is the i -th singular value of $\bar{X}^{(k)}$.

Because of the truncation strategy [28]–[30] and the property of the penalty function, the proposed model preserves the first r principal singular values and adaptively penalizes that in the $[r+1, h]$ range. This effectively mitigates the over-shrinkage drawback of convex surrogate functions such as TNN, thus enabling better capturing of the low-rank structure of a tensor.

Therefore, we obtained the following truncated nonconvex fraction function minimization (TNFFM) problem:

$$\arg \min_{\mathcal{X}} \mathcal{P}_{a,r}(\mathcal{X}) + \frac{1}{2} \|\mathcal{X} - \mathcal{A}\|_F^2. \quad (5)$$

To solve this problem, we first introduce the following lemma and theorem, which are used to solve the matrix and tensor problems related to the proposed penalty function.

Lemma 1. [31] For matrix $Y \in \mathbb{R}^{m \times n}$, its SVD is $Y = U \cdot \text{Diag}(\sigma(Y)) \cdot V^\top$. Then the solution to the following minimization problem

$$\arg \min_X P_a(X) + \frac{\lambda}{2} \|X - Y\|_F^2 \quad (6)$$

is given by

$$X^* = U \cdot D_{a,\lambda}(Y) \cdot V^\top, \quad (7)$$

where $D_{a,\lambda}(Y) = \text{diag}(\gamma_1, \gamma_2, \dots, \gamma_h)$, $\gamma_i = \text{prox}_{a,\lambda}(\sigma_i(Y))$. The proximal operator $\text{prox}_{a,\lambda}(x) := \arg \min_{\sigma} \rho_a(\sigma) + \frac{\lambda}{2}(\sigma - x)^2$ is defined as

$$\text{prox}_{a,\lambda}(x) = \begin{cases} g_{a,\lambda}(x), & |x| > t_{a,\lambda}, \\ 0, & |x| \leq t_{a,\lambda}, \end{cases} \quad (8)$$

where

$$t_{a,\lambda} = \begin{cases} a/\lambda, & \lambda \geq 2a^2, \\ \sqrt{2/\lambda} - 1/(2a), & \lambda < 2a^2, \end{cases} \quad (9)$$

and

$$g_{a,\lambda}(x) = \text{sign}(x) \left(\frac{\frac{1+a|x|}{3} \left(1 + 2 \cos \left(\frac{\varphi(x)}{3} - \frac{\pi}{3} \right) \right) - 1}{a} \right), \quad (10)$$

with

$$\varphi(x) = \arccos \left(\frac{27a^2}{2\lambda(1+a|x|)^3} - 1 \right). \quad (11)$$

Theorem 1. Let $\mathcal{A} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$ be a tensor with $h = \min(n_1, n_2)$, and its T-SVD is $\mathcal{A} = \mathcal{U} * \mathcal{S} * \mathcal{V}^\top$. Then, the closed-form solution to the following minimization problem

$$\arg \min_{\mathcal{X}} \mathcal{P}_{a,r}(\mathcal{X}) + \frac{\lambda}{2} \|\mathcal{X} - \mathcal{A}\|_F^2, \quad (12)$$

is given by

$$\mathcal{X}^* = \mathcal{U} * \text{ifft}(\bar{\mathcal{B}}) * \mathcal{V}^\top, \quad (13)$$

where $\bar{\mathcal{B}}$ is a tensor whose k -th frontal slice is $\bar{B}^{(k)} = \text{Diag}(\varrho^*)$, and

$$\varrho^* = \begin{cases} \sigma_i(\bar{A}^{(k)}), & 0 \leq i < r+1, \\ D_{a,\lambda}(R), & r+1 \leq i \leq h, \end{cases} \quad (14)$$

with $R = [0, \dots, 0, \sigma_{r+1}(\bar{A}^{(k)}), \dots, \sigma_h(\bar{A}^{(k)})]^\top$.

Proof. In the Fourier domain, combining the definition of $\mathcal{P}_{a,r}$ and the fact that $\|\mathcal{A}\|_F^2 = \frac{1}{n_3} \|\bar{\mathcal{A}}\|_F^2$, the model (12) can be rewritten as

$$\arg \min_{\bar{\mathcal{X}}^{(k)}} \sum_{k=1}^{n_3} \left(P_{a,r}(\bar{\mathcal{X}}^{(k)}) + \frac{\lambda}{2} \|\bar{\mathcal{X}}^{(k)} - \bar{\mathcal{A}}^{(k)}\|_F^2 \right). \quad (15)$$

According to the Von Neumann's trace inequality [32], we can obtain

$$\begin{aligned} & \|\bar{\mathcal{X}}^{(k)} - \bar{\mathcal{A}}^{(k)}\|_F^2 \\ &= \sum_{i=1}^h \sigma_i^2(\bar{\mathcal{X}}^{(k)}) - 2 \text{tr} \left((\bar{\mathcal{X}}^{(k)})^\top \bar{\mathcal{A}}^{(k)} \right) + \sum_{i=1}^h \sigma_i^2(\bar{\mathcal{A}}^{(k)}) \\ &\geq \sum_{i=1}^h \sigma_i^2(\bar{\mathcal{X}}^{(k)}) - 2 \sum_{i=1}^h \left(\sigma_i(\bar{\mathcal{X}}^{(k)}) \sigma_i(\bar{\mathcal{A}}^{(k)}) \right) + \sum_{i=1}^h \sigma_i^2(\bar{\mathcal{A}}^{(k)}) \\ &= \sum_{i=1}^h \left(\sigma_i(\bar{\mathcal{X}}^{(k)}) - \sigma_i(\bar{\mathcal{A}}^{(k)}) \right)^2. \end{aligned} \quad (16)$$

Furthermore, (15) can be reformulated as

$$\arg \min_{\sigma_i(\bar{\mathcal{X}}^{(k)})} \sum_{k=1}^{n_3} \sum_{i=1}^r \left(\frac{\lambda}{2} \left(\sigma_i(\bar{\mathcal{X}}^{(k)}) - \sigma_i(\bar{\mathcal{A}}^{(k)}) \right)^2 \right) + \sum_{k=1}^{n_3} \sum_{i=r+1}^h \left(\rho_a(\sigma_i(\bar{\mathcal{X}}^{(k)})) + \frac{\lambda}{2} \left(\sigma_i(\bar{\mathcal{X}}^{(k)}) - \sigma_i(\bar{\mathcal{A}}^{(k)}) \right)^2 \right). \quad (17)$$

Therefore, the original problem (15) is equivalently transformed into a combination of independent quadratic equations for each $\sigma_i(\bar{\mathcal{X}}^{(k)})$. Let $F(\sigma(\bar{\mathcal{X}}^{(k)}))$ denote the objective function of (17). Then, the minimum of F by ϱ_i^* is given by

$$\frac{\partial F}{\partial \sigma_i(\bar{\mathcal{X}}^{(k)})} = 0. \quad (18)$$

When $0 \leq i < r+1$, the solution of (17) is

$$\varrho_i^* = \sigma_i(\bar{\mathcal{A}}^{(k)}). \quad (19)$$

When $r+1 \leq i \leq h$, according to the Lemma 1, the solution of (17) is

$$\varrho_i^* = D_{a,\lambda}(R), \quad (20)$$

where $R = [0, \dots, 0, \sigma_{r+1}(\bar{\mathcal{A}}^{(k)}), \dots, \sigma_h(\bar{\mathcal{A}}^{(k)})]^\top$. Combining (19) and (20), we further obtain

$$\varrho^* = \begin{cases} \sigma_i(\bar{\mathcal{A}}^{(k)}), & 0 \leq i < r+1, \\ D_{a,\lambda}(R), & r+1 \leq i \leq h, \end{cases} \quad (21)$$

Let $\bar{\mathcal{B}}^{(k)} = \text{Diag}(\varrho^*)$ be the k -th frontal slice of tensor $\bar{\mathcal{B}}$, then the optimal solution of (12) is

$$\mathcal{X}^* = \mathcal{U} * \text{ifft}(\bar{\mathcal{B}}) * \mathcal{V}^\top, \quad (22)$$

where $\mathcal{U} = \text{ifft}(\bar{\mathcal{U}}, [], 3)$, $\mathcal{V} = \text{ifft}(\bar{\mathcal{V}}, [], 3)$. $\square$

The whole optimization process of model (3) is summarized in Algorithm 1.

Algorithm 1 Solving the TNFFM Problem.

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1: Input:  $\mathcal{A} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$ ;
2:  $\bar{\mathcal{A}} = \text{fft}(\mathcal{A}, [], 3)$ ;
3: for  $k = 1, 2, \dots, n_3$  do
4:    $[U, S, V] = \text{svd}(\bar{\mathcal{A}}(:, :, k))$ ;
5:    $\bar{\mathcal{U}}^{(k)} = U, \bar{\mathcal{S}}^{(k)} = S, \bar{\mathcal{V}}^{(k)} = V$ ;
6: end for
7: for  $k = 1, 2, \dots, n_3$  do
8:   Update  $\varrho^*$  by (21);
9:    $\bar{\mathcal{B}}^{(k)} = \text{Diag}(\varrho^*)$ ;
10: end for
11:  $\mathcal{U} = \text{ifft}(\bar{\mathcal{U}}, [], 3), \mathcal{S} = \text{ifft}(\bar{\mathcal{S}}, [], 3), \mathcal{V} = \text{ifft}(\bar{\mathcal{V}}, [], 3)$ ;
12: Output:  $\mathcal{X} = \mathcal{U} * \mathcal{S} * \mathcal{V}^\top$ .
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III. TRPCA WITH TRUNCATED NONCONVEX FRACTION FUNCTION MINIMIZATION

A. The TRPCA Model

As mentioned in the introduction, solving the tensor rank minimization problem is extremely difficult. Most existing methods employ convex surrogate functions for rank approximation, which often leads to biased and unstable rank estimation. This is because there are significant differences among the non-zero singular values of the tensor, which are crucial for accurate tensor recovery. However, the uniform shrinkage of singular values imposed by convex methods tends to over-penalize the dominant singular values, thereby degrading the recovery performance.

To address this issue, we propose a novel TRPCA model based on the above truncated nonconvex fraction penalty function, which can be expressed as

$$\min_{\mathcal{X}, \mathcal{E}} \mathcal{P}_{a,r}(\mathcal{X}) + \lambda \|\mathcal{E}\|_1 \quad \text{s.t. } \mathcal{D} = \mathcal{X} + \mathcal{E}. \quad (23)$$

Benefiting from the introduction of the ℓ_1 -norm and the unbiased $\mathcal{P}_{a,r}$ function, model (23) can separate low-rank and sparse components more accurately. Specifically, the proposed model can retain larger singular values and impose stronger penalties on smaller singular values, thus resulting in a more reliable low-rank approximation.

B. Optimization Scheme

Generally, the ADMM framework is applied to solve the optimization problem (23), whose augmented Lagrangian function is

$$\begin{aligned} \mathcal{L}(\mathcal{X}, \mathcal{E}, \mathcal{Y}, \mu) &= \mathcal{P}_{a,r}(\mathcal{X}) + \lambda \|\mathcal{E}\|_1 + \langle \mathcal{Y}, \mathcal{D} - \mathcal{X} - \mathcal{E} \rangle \\ &\quad + \frac{\mu}{2} \|\mathcal{D} - \mathcal{X} - \mathcal{E}\|_F^2, \end{aligned} \quad (24)$$

Where $\mathcal{Y}$ and μ are the dual variable and positive penalty parameter, respectively. Therefore, we can update the variables alternately by solving the following subproblems.

1) **Update** $\mathcal{X}$: $\mathcal{X}^{k+1}$ can be updated as follows:

$$\mathcal{X}^{k+1} = \arg \min_{\mathcal{X}} \mathcal{P}_{a,r}(\mathcal{X}) + \frac{\mu^k}{2} \|\mathcal{X} - (\mathcal{D} + \frac{\mathcal{Y}^k}{\mu^k} - \mathcal{E}^k)\|_F^2. \quad (25)$$

Algorithm 2 ADMM for TRPCA via the TNFFM.

- 1: **Input:** $\mathcal{D}$, and parameters λ, a ;
 - 2: **Initialize:** $\mathcal{X} = \mathcal{D}, \mathcal{E} = \mathcal{Y} = 0, \rho = 1.1, \mu = 1e-4, tol = 1e-8, k = 0, \text{ and } L = 500$;
 - 3: **While** not converged **do**
 - 4: Update $\mathcal{X}$ by (26);
 - 5: Update $\mathcal{E}$ by (29);
 - 6: Update $\mathcal{Y}$ and μ by (30);
 - 7: $k \leftarrow k + 1$;
 - 8: Check the convergence condition:
 - 9: $\|\mathcal{D} - \mathcal{E}^{k+1} - \mathcal{X}^{k+1}\|_F / \|\mathcal{D}\|_F < tol$ or $k > L$;
 - 10: **end while**
 - 11: **Output:** The recovery tensor $\mathcal{X}$ and sparse tensor $\mathcal{E}$.
-

Let $\mathcal{C} = \mathcal{D} + \mathcal{Y}^k / \mu^k - \mathcal{E}^k$, and according to Theorem 1, we can obtain the following optimal solution:

$$\mathcal{X}^{k+1} = \mathcal{U} * \text{ifft}(\overline{\mathcal{B}}) * \mathcal{V}^\top, \quad (26)$$

where $\overline{\mathcal{B}}$ is a tensor whose k -th frontal slice is $\overline{B}^{(k)} = \text{Diag}(\varrho^*)$, and

$$\varrho^* = \begin{cases} \sigma_i(\overline{C}^{(k)}), & 0 \leq i < r + 1, \\ D_{a,\lambda}(R), & r + 1 \leq i \leq h, \end{cases} \quad (27)$$

with $R = [0, \dots, 0, \sigma_{r+1}(\overline{C}^{(k)}), \dots, \sigma_h(\overline{C}^{(k)})]^\top$.

2) **Update $\mathcal{E}$:** $\mathcal{E}^{k+1}$ can be updated as follows:

$$\mathcal{E}^{k+1} = \arg \min_{\mathcal{E}} \lambda \|\mathcal{E}\|_1 + \frac{\mu^k}{2} \|\mathcal{E} - (\mathcal{D} + \frac{\mathcal{Y}^k}{\mu^k} - \mathcal{X}^{k+1})\|_F^2. \quad (28)$$

Let $\mathcal{M} = (\mathcal{D} + \mathcal{Y}^k / \mu^k - \mathcal{X}^{k+1})$, then the solution to (28) can be obtained by the following soft-thresholding operator

$$\mathcal{E}^{k+1} = \mathcal{S}_{\lambda/\mu^k}(\mathcal{M}), \quad (29)$$

where $\mathcal{S}_{\lambda/\mu^k}(\mathcal{M}_{ijk}) = \text{sign}(\mathcal{M}_{ijk}) \max(|\mathcal{M}_{ijk}| - \lambda/\mu^k, 0)$.

3) **Update $\mathcal{Y}$ and μ :** $\mathcal{Y}$ and μ can be updated as follows:

$$\begin{aligned} \mathcal{Y}^{k+1} &= \mathcal{Y}^k + \mu^k (\mathcal{D} - \mathcal{E}^{k+1} - \mathcal{X}^{k+1}), \\ \mu^{k+1} &= \min(\rho \mu^k, \mu^{\max}), \end{aligned} \quad (30)$$

where $\rho > 1$ is a step parameter, and $\mu^{\max}$ is the upper bound of the parameter μ .

Thus, the overall optimization scheme for model (23) based on ADMM is summarized in Algorithm 2.

C. Complexity Analysis

We briefly analyze the computational complexity of the proposed method. For a tensor $\mathcal{X} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$ with $h = \min(n_1, n_2)$, the computational complexity of Algorithm 1 mainly depends on the FFT and T -SVD operations, leading to an overall complexity of $O(n_1 n_2 n_3 \log n_3 + n_1 n_2 n_3 h)$. The computational complexity of each iteration of Algorithm 2 mainly depends on the update of tensor $\mathcal{X}$ and the element-wise tensor operations of other variables, which are the complexity of Algorithm 1 and $O(n_1 n_2 n_3)$, respectively. Therefore, the overall complexity of each iteration is $O(n_1 n_2 n_3 (\log n_3 + h))$.

TABLE I

THE RELATIVE ERROR RESULTS OF SYNTHESIZED TENSORS WITH $pr \in [0.05 : 0.05 : 0.2]$ AND $ps = 0.05$.

Rank	Index	TRPCA	PSTNN	ETRPCA	t- $S_{w,p}$	SRPCA	Ours
10	$\frac{\ \hat{\mathcal{X}} - \mathcal{X}\ _F}{\ \mathcal{X}\ _F}$	3.71e-09	1.76e-08	7.04e-09	1.99e-08	1.67e-08	1.22e-09
	$\frac{\ \hat{\mathcal{E}} - \mathcal{E}\ _F}{\ \mathcal{E}\ _F}$	2.07e-10	5.96e-10	6.04e-10	1.88e-09	7.50e-09	2.22e-10
	Time	13.10	50.64	23.67	8.69	20.99	20.31
20	$\frac{\ \hat{\mathcal{X}} - \mathcal{X}\ _F}{\ \mathcal{X}\ _F}$	2.24e-09	1.07e-08	4.79e-09	1.13e-08	1.22e-08	1.51e-09
	$\frac{\ \hat{\mathcal{E}} - \mathcal{E}\ _F}{\ \mathcal{E}\ _F}$	7.07e-10	3.56e-09	1.58e-09	1.69e-09	7.46e-09	4.76e-10
	Time	12.85	34.18	29.16	9.56	24.40	21.74
30	$\frac{\ \hat{\mathcal{X}} - \mathcal{X}\ _F}{\ \mathcal{X}\ _F}$	3.81e-09	1.73e-08	1.38e-09	9.32e-09	1.46e-08	1.71e-09
	$\frac{\ \hat{\mathcal{E}} - \mathcal{E}\ _F}{\ \mathcal{E}\ _F}$	1.75e-09	7.70e-09	6.17e-10	1.86e-09	1.07e-08	9.21e-10
	Time	12.90	45.95	32.67	10.78	27.67	22.83
40	$\frac{\ \hat{\mathcal{X}} - \mathcal{X}\ _F}{\ \mathcal{X}\ _F}$	3.26e-09	1.76e-08	4.62e-09	3.83e-09	1.72e-08	2.10e-09
	$\frac{\ \hat{\mathcal{E}} - \mathcal{E}\ _F}{\ \mathcal{E}\ _F}$	1.90e-09	9.95e-09	2.64e-09	1.74e-09	1.42e-08	9.97e-10
	Time	16.65	46.37	35.88	11.85	31.58	26.48

IV. EXPERIMENTS

In this section, the effectiveness of the proposed method is verified through numerical experiments on synthetic experiments, color image denoising and video background subtraction. We compared our proposed method with five state-of-the-art methods, including TRPCA [7], PSTNN [21], ETRPCA [22], t- $S_{w,p}$ [33] and SRPCA [34]. In the following experiments, the best and second-best results for all reported numerical results are highlighted in red and blue, respectively. Furthermore, all experiments are performed on a Windows 10 computer with an Intel Core i5-10400 CPU (2.90 GHz), 16 GB of RAM, using MATLAB R2021b.

A. Experimental Settings

The performance of different methods is evaluated in terms of running time and relative squared error (RSE), structural similarity index (SSIM) [35], peak signal-to-noise ratio (PSNR), and running time (in seconds). Generally, RSE is used to calculate the difference between two tensors in simulation experiments, while PSNR and SSIM are used to quantitatively evaluate the performance of image restoration and the ability to preserve the visual effects of the image, respectively. Higher values for PSNR and SSIM, and lower values for RSE and running time, respectively indicate better recovery performance.

In addition, the parameter settings for all methods in this experiment are summarized as follows. Specifically, for TRPCA, λ is set within $\{0.003, 0.01, 0.02\}$. For PSTNN, λ is set to $1/(k\sqrt{\max(n_1, n_2)n_3})$ with $k \in [0.1 : 0.1 : 2]$. For the proposed method, $\alpha \in [0.03 : 0.005 : 2]$, $\lambda \in \{0.0005, 0.001\}$, and $r \in [1 : 1 : 10]$, with the best result reported. For all other competing methods, the default parameters recommended by the authors are used without further tuning.

B. Synthetic Experiments

In this experiment, tensors $\mathcal{D} \in \mathbb{R}^{n \times n \times m}$ with $n = 200$ and $m = 30$ are considered, where the rank is constrained by pr and the

TABLE II

THE RELATIVE SQUARED ERROR RESULTS OF SYNTHESIZED TENSORS WITH $pr \in [0.05 : 0.05 : 0.2]$ AND $ps = 0.1$.

Rank	Index	TRPCA	PSTNN	ETRPCA	t- $S_{w,p}$	SRPCA	Ours
10	$\frac{\ \hat{\mathcal{X}} - \mathcal{X}\ _F}{\ \mathcal{X}\ _F}$	7.16e-09	6.22e-08	5.84e-09	2.28e-08	2.83e-08	1.53e-09
	$\frac{\ \hat{\mathcal{E}} - \mathcal{E}\ _F}{\ \mathcal{E}\ _F}$	8.39e-10	4.40e-09	8.19e-10	2.18e-09	8.83e-09	2.45e-10
	Time	11.91	30.24	23.47	9.22	23.43	18.52
20	$\frac{\ \hat{\mathcal{X}} - \mathcal{X}\ _F}{\ \mathcal{X}\ _F}$	5.00e-09	7.97e-08	7.40e-09	1.21e-08	2.82e-08	3.74e-09
	$\frac{\ \hat{\mathcal{E}} - \mathcal{E}\ _F}{\ \mathcal{E}\ _F}$	1.31e-09	2.00e-08	1.91e-09	1.83e-09	1.20e-08	1.08e-09
	Time	13.16	45.20	33.53	10.03	27.59	21.416
30	$\frac{\ \hat{\mathcal{X}} - \mathcal{X}\ _F}{\ \mathcal{X}\ _F}$	7.50e-09	3.58e-08	3.85e-09	4.91e-09	2.54e-08	3.53e-09
	$\frac{\ \hat{\mathcal{E}} - \mathcal{E}\ _F}{\ \mathcal{E}\ _F}$	2.68e-09	1.26e-08	1.27e-09	1.65e-09	1.28e-08	1.38e-09
	Time	15.66	47.03	38.82	11.84	32.97	22.89
40	$\frac{\ \hat{\mathcal{X}} - \mathcal{X}\ _F}{\ \mathcal{X}\ _F}$	7.58e-09	3.61e-08	6.80e-09	5.91e-09	3.30e-08	4.04e-09
	$\frac{\ \hat{\mathcal{E}} - \mathcal{E}\ _F}{\ \mathcal{E}\ _F}$	3.37e-09	1.56e-08	3.05e-09	2.64e-09	1.91e-08	1.67e-09
	Time	18.89	49.28	47.25	15.40	38.66	27.43

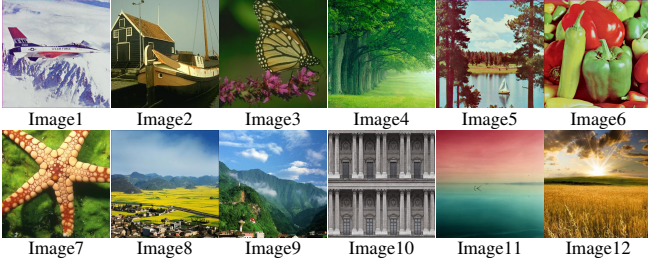


Fig. 1: The original test images.

sparsity is represented by ps . The observation tensor $\mathcal{D}$ consists of the low-rank component $\mathcal{X}$ and the sparse component $\mathcal{E}$, and is expressed as $\mathcal{D} = \mathcal{X} + \mathcal{E}$. We first generate a tensor $\mathcal{X} \in \mathbb{R}^{n \times n \times m}$ as $\mathcal{X} = \mathcal{P} * \mathcal{Q}^\top$, where $\mathcal{P} \in \mathbb{R}^{n \times r \times m}$ and $\mathcal{Q} \in \mathbb{R}^{r \times n \times m}$, whose entries are independently sampled from $\mathcal{N}(1, 1/n)$ distribution, with $r = n * pr$. Then, we generate the sparse tensor $\mathcal{E}$ with $ps \times n^2 \times m$ nonzero entries. To investigate the influence of the rank of $\mathcal{X}$ and the sparsity on the effectiveness of the algorithm, we set $pr \in [0.05 : 0.05 : 0.2]$ and $ps \in \{0.05, 0.1\}$ in the following experiments.

The experimental results on the synthetic data are summarized in Tables I and II. As shown in these tables, the proposed method achieves the lowest RSE values in almost all experimental settings. Despite the increasing tensor rank and sparsity, the proposed method consistently maintains a very small RSE (below 10^{-8}), demonstrating its strong capability in accurately recovering low-rank and sparse tensors. In terms of computational efficiency, the proposed method is on average approximately twice as fast as the PSTNN method. Overall, compared with other methods, the proposed approach achieves more accurate and efficient tensor recovery under varying tensor ranks and sparse ratios.

C. Color Image Denoising

In this experiment, twelve commonly used and well-known real color images were used to verify the performance of

TABLE III

THE AVERAGE RECOVERY RESULTS OF ALL COLOR IMAGES WITH VARIOUS NOISE CONDITIONS.

NL	Index	TRPCA	PSTNN	ETRPCA	t- $S_{w,p}$	SRPCA	Ours
0.05	PSNR	27.74	27.82	27.93	27.57	27.61	29.02
	SSIM	0.8999	0.9007	0.9131	0.8797	0.9008	0.9115
	Time	7.52	23.19	23.68	9.08	15.30	16.30
0.1	PSNR	23.60	23.76	24.58	21.84	23.72	25.52
	SSIM	0.7787	0.7881	0.8187	0.7107	0.7854	0.8500
	Time	7.16	23.20	23.61	8.89	15.22	19.04
0.15	PSNR	20.79	21.10	21.89	18.35	20.97	23.58
	SSIM	0.6671	0.6854	0.7165	0.5620	0.6785	0.7810
	Time	7.08	23.39	23.23	8.94	15.36	19.36
0.2	PSNR	18.69	19.13	19.69	15.85	18.90	21.99
	SSIM	0.5699	0.6002	0.6174	0.4406	0.5850	0.7234
	Time	7.10	22.95	23.52	9.04	15.41	19.23

TABLE IV

DETAILS OF EACH VIDEO USED IN OUR EXPERIMENT.

	Hall	Escalator	WaterSurface	Lobby	ShoppingMall
Sizes	144 × 176	130 × 160	128 × 160	128 × 160	256 × 320
frames	280	200	190	170	265

the proposed method, as shown in Fig. 1, which are of size $300 \times 300 \times 3$ or $512 \times 512 \times 3$. For each image, a subset of pixels are randomly set to random values within the range of $[0, 255]$. In this experiment, the noise levels (NL) were set within $\{0.05, 0.1, 0.15, 0.2\}$.

The average denoising results of twelve color images at different noise levels are presented in Table III. As can be seen, the proposed method consistently outperforms the other competing approaches in terms of denoising performance. In particular, our method achieves the highest average PSNR and SSIM values at almost all noise levels, demonstrating its superior denoising capability. Even at a high noise level of 0.2, our proposed method still achieves satisfactory results, achieving the highest average PSNR of 21.99 dB and SSIM of 0.72, which are approximately 1.50 dB and 0.05 higher than those of the second-best ETRPCA method, respectively. In addition, the proposed method attains competitive computational efficiency while maintaining the best PSNR and SSIM values. On average, our method runs approximately 20% faster than PSTNN and ETRPCA.

The visual results of denoising some color images are shown in Fig. 2. From a global view, our method produced the best visual results. Specifically, the images obtained using the ETRPCA, t- $S_{w,p}$, and SRPCA methods still retain some noise, while the images obtained using the TRPCA and PSTNN methods are overly smooth and lose many image texture details. In contrast, it can be observed that the images produced by the proposed method are closest to the ground truth, preserving more realistic textures and colors. Overall, these visual comparisons further demonstrate the superiority of the proposed method in denoising performance.

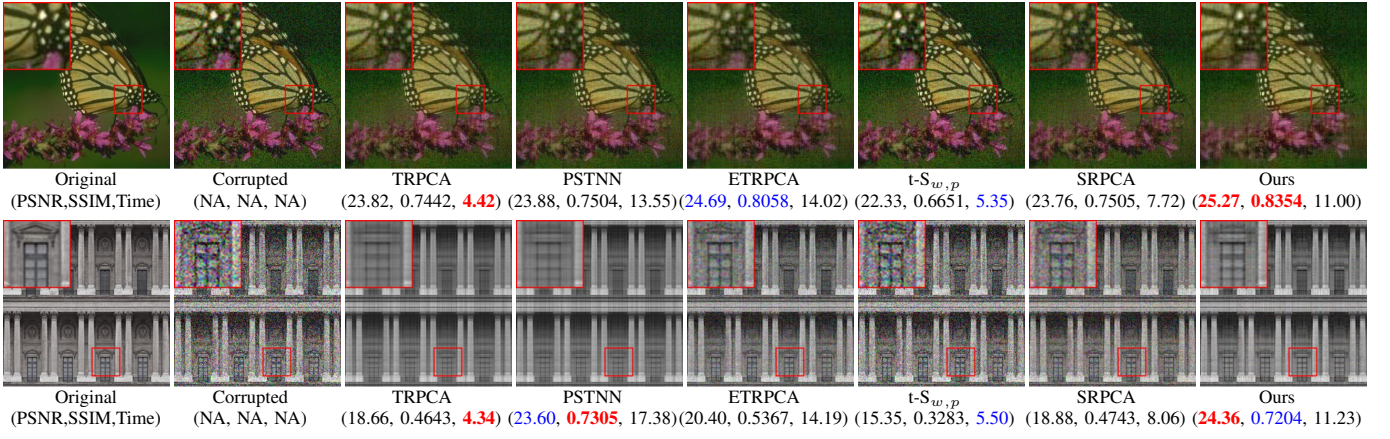


Fig. 2: The experimental results of color image denoising for Image3 at NL = 0.1 and Image10 at NL = 0.2.

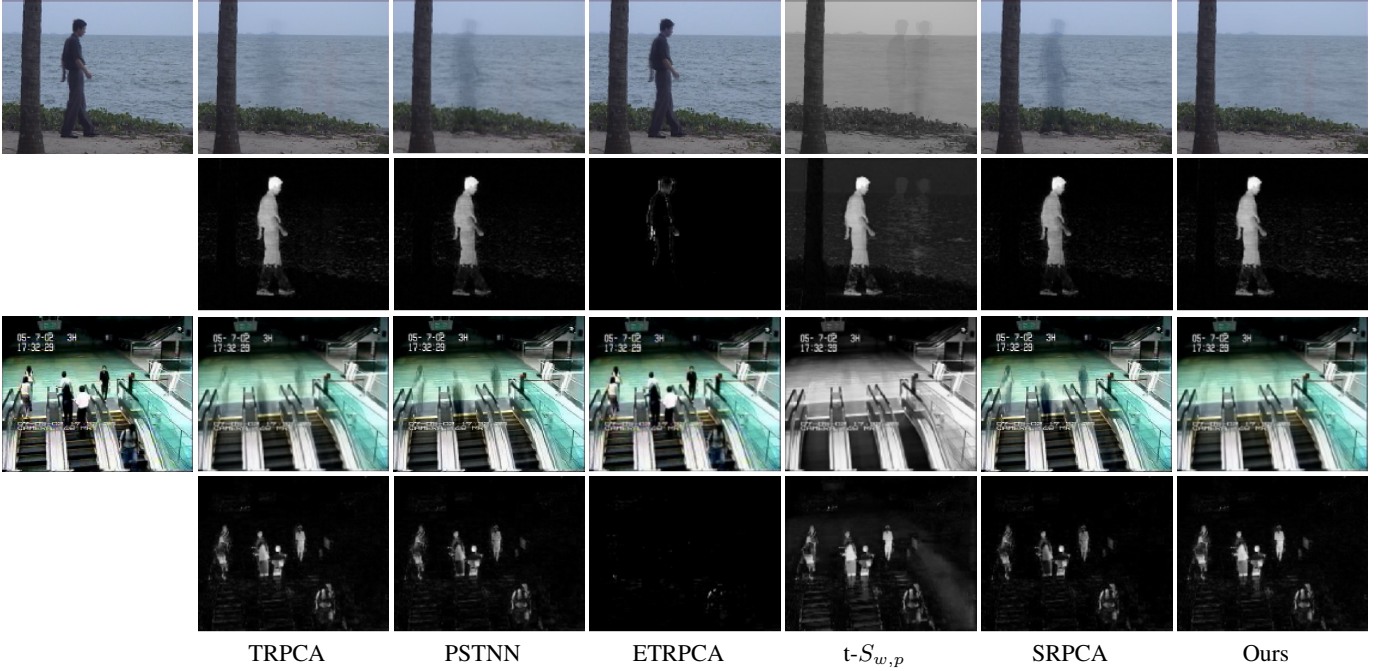


Fig. 3: The visual effects of a single frame after background subtraction on the video WaterSurface and Escalator.

D. Video Background Subtraction

In this subsection, the video background is highly correlated and can therefore be regarded as a low-rank component. Conversely, the moving foreground occupies only a small portion of all image frames and can therefore be considered a sparse component. Therefore, the video background subtraction problem can be viewed as a tensor decomposition problem. In this experiment, we obtained video data from the SBI dataset [36] and the I2R dataset [37]. These datasets include video footage from five indoor and outdoor environments, including Hall, Escalator, WaterSurface, Lobby and ShoppingMall. The detailed information for each video is shown in Table IV.

The visual results of the background subtraction experiments are shown in Fig. 3. As can be observed, our method achieves the best visual performance across all test videos. In

contrast, the ETRPCA method exhibits the poorest performance, recovering only a coarse silhouette of the foreground object. Moreover, the PSTNN and SRPCA methods still suffer from severe human ghosting, while the background image restored by the $t-S_{w,p}$ method loses its true colors. From a global view, our method produces the least amount of residual foreground objects in the recovered background while preserving the true background colors. Overall, the visual comparison of background subtraction results indicate that the proposed method can more accurately separate low-rank and sparse components.

V. CONCLUSION

In this paper, we proposed a truncated nonconvex fractional function as a tighter approximation to the tensor multi-rank

and formulated a TRPCA model based on it. Specifically, this function facilitates the separation of low-rank and sparse components by effectively suppressing small singular values while preserving the dominant ones. Moreover, the proximal mapping of the proposed function admits a closed-form solution, which significantly improves both the accuracy and efficiency of the optimization process. Subsequently, an efficient algorithm based on the ADMM framework was developed to solve the resulting optimization problem. Extensive experimental results demonstrate that the proposed method achieves superior performance in sparse noise separation compared to several state-of-the-art methods.

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