GBFRVFL: Granular-Ball Computing-Based Fuzzy Random Vector Functional Link Network

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***Abstract*—In practical machine learning tasks, data are often contaminated with noise, outliers, and class imbalance, which can degrade the performance of conventional models. While random vector functional link (RVFL) networks offer fast training and strong generalization, they do not explicitly handle uncertainty or exploit local data structure. To address these limitations, we propose a fuzzy granular-ball random vector functional link (GBFRVFL) framework that leverages granular-ball computing to abstract raw samples into adaptive granular balls. Within this framework, we introduce two membership assignment schemes: (i) F-GBRVFL, which incorporates fuzzy membership to quantify the reliability of each granular ball, and (ii) SDAP-GBRVFL, which we propose, incorporates a novel statistical density-adaptive pythagorean membership (SDAPM) scheme that dynamically adjusts membership and non-membership values based on class variance, local sparsity, and granular-ball compactness. These schemes enhance robustness to noise, outliers, class imbalance, and uncertainty in granular-ball distributions, while retaining the computational efficiency of RVFL networks. Extensive experiments on 37 benchmark UCI and KEEL datasets under both clean and noisy conditions demonstrate that the proposed models consistently outperform baseline models, achieving superior accuracy and stability. The results validate the effectiveness of integrating granular-ball computing with adaptive membership schemes for reliable, scalable, and noise-tolerant learning. The code and supplementary material of the paper can be accessed using the following link: https://github.com/mtanveer1/GBFRVFL.**

***Index Terms*—Granular ball, Granular computing, Random Vector Functional Link (RVFL), Noise.**

I. Introduction

ARtificial neural networks (ANNs), including single hidden layer feedforward neural (SLFN) architectures, have been widely applied to classification and regression tasks [1, 2], owing to their strong universal function approximation properties [3]. A traditional approach for training SLFNs is the backpropagation (BP) algorithm [4]; however, it often encounters issues such as slow convergence, susceptibility to local minima, and sensitivity to the choice of learning rate. To address these limitations, several learning models based on randomization have been developed, including the Schmidt network [5], random vector functional link (RVFL) neural networks [6], extreme learning machines (ELM) [7], and radial basis function networks (RBFN) [8]. Among these, the RVFL network has a simple architecture and efficient performance. The RVFL training procedure is carried out in two distinct phases. Initially, the weights and biases connecting the input layer to the hidden layer are randomly assigned within a predefined range [9] and remain unchanged during training. Subsequently, the output-layer parameters are computed analytically using a closed-form solution, enabling efficient and stable learning [10]. Due to its fast training speed and strong generalization capability, the RVFL network has been successfully applied in a wide range of domains, including time-series prediction [11], hyperspectral image classification [12], prediction of DNA-binding proteins [13], and nonlinear system identification [14].

Despite their efficiency, RVFL models and their variants often degrade in performance on noisy or imperfect datasets because they assign equal importance to all training samples, making them sensitive to noise, outliers, and class imbalance. Fuzzy theory has been widely used to mitigate such issues [15, 16]. Granular computing (GC) provides a framework for handling uncertainty and incomplete data through information granules, as introduced by Zadeh [17]. Significant advances have been made by integrating GC with machine learning techniques such as rough sets, fuzzy sets, three-way decision models, and formal concept analysis [18, 19]. Inspired by the global precedence characteristic of human cognition [20], multigranularity frameworks have been developed to enhance uncertainty handling [21]. Recent works include the granular-ball twin support vector machine (GBTSVM) [22, 23] and granular ball RVFL (GB-RVFL) [24], which use granular balls instead of individual samples. While these approaches improve performance on complex, high-dimensional data, they remain vulnerable to noise and outliers that can distort granular

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representations and reduce classification reliability.

To address the challenges posed by noise, outliers, and uncertainty within granular balls, we developed a novel membership assignment mechanism, termed statistical density-adaptive pythagorean membership (SDAPM), and incorporated it into the GBRVFL network, resulting in the proposed SDAP-GBRVFL model. In addition, a fuzzy membership scheme is also integrated in the GBRVFL framework to call F-GBRVFL. Unlike conventional granular-ball classifiers, which often rely on global thresholds or fixed radii and are therefore highly sensitive to extreme samples, the SDAPM scheme adaptively adjusts membership degrees based on local statistical properties, including class-wise variance and neighborhood sparsity. By integrating this adaptive membership into the GBRVFL framework, the model leverages GBRVFL's simple architecture and fast training while benefiting from the coarse-grained, noise-resilient representation of granular balls.

The paper's key highlights are as follows:

1) We propose a density-adaptive Pythagorean membership scheme that dynamically adjusts membership values using class variance, sparsity, and compactness to handle noise, outliers, and class imbalance.
2) We propose F-GBRVFL and SDAP-GBRVFL, which use fuzzy and density-adaptive membership schemes to improve robustness to noise, outliers, class imbalance, and uncertain granular-ball distributions.
3) Using granular balls instead of individual samples reduces computation by inverting only the GB-center matrix, while preserving discriminative information and reducing noise and outlier effects.
4) F-GBRVFL and SDAP-GBRVFL are evaluated on KEEL and UCI benchmark datasets under clean and noisy labels and compared with state-of-the-art models.

II. Related Work

In this section, we provide an overview and discussion of the granular ball computing (GBC) methodology.

A. Granular-Ball Computing

The core idea of GBC is to represent the original dataset using a collection of granular balls, where each granular ball summarizes a group of samples. Let $\mathcal{M} = \{(x_1, l_1), (x_2, l_2), \ldots, (x_n, l_n)\}$ denote a dataset, where $X = \{x_j \mid x_j \in \mathbb{R}^d, , j = 1, 2, \ldots, n\}$ represents the feature matrix consisting of d attributes, and $L = \{l_j \mid l_j \in \mathbb{R}, j = 1, 2, \ldots, n\}$ denotes the associated label set. The entire sample space $\mathcal{D}$ is then partitioned or covered by a collection of granular balls $\mathcal{GB} = \{GB_i \mid i = 1, 2, \ldots, k\}$. For the i^{th} granular-ball GB_i, its center is defined as $c_i = \frac{1}{n_i}\sum_{k=1}^{n_i} x_{ij}$, where x_{ij} denotes the j^{th} sample belonging to GB_i, and n_i is the total number of samples contained in this granular-ball. The radius r_i of GB_i can be computed using the average-distance, $r_i = \frac{1}{n_i}\sum_{k=1}^{n_i} |x_{ij} - c_i|$. To reduce the influence of noisy samples inside a granular ball, a representative label is assigned to each GB_i based on majority voting. Specifically, the label y_i is determined as the class that occurs most frequently among the samples contained in GB_i. The purity of the granular ball denoted by p_i is then defined as the ratio between the number of samples in GB_i that share the dominant label y_i and the total number of samples n_i within the granular ball.

Within the GBC framework, the first step is to construct granular balls that represent the entire dataset. This is achieved by generating granular balls through the following optimization problem:

$$\begin{aligned} &\min \ \gamma_1 \sum_{GB_i \in \mathcal{GB}} |GB_i| + \gamma_2 k, \\ &\text{s.t quality}(GB_i) \geq \rho, \end{aligned} \tag{1}$$

where γ_1 and γ_2 are weighting parameters, k denotes the total number of granular balls, and quality(GB_i) measures the fraction of samples sharing the dominant label within the granular ball GB_i.

III. The Proposed Fuzzy Granular-Ball Random Vector Functional Link Network

In real-world data, noise, outliers, and class overlap often reduce model reliability. Although RVFL networks are fast and generalize well, they do not explicitly capture uncertainty or local relationships. Granular-ball computing addresses this by grouping similar samples into adaptive granular balls, while fuzzy theory enhances robustness by assigning membership degrees that reflect granule reliability. Building on this, we propose two variants: F-GBRVFL, which uses a fuzzy membership scheme to handle noisy and ambiguous samples, and SDAP-GBRVFL, which incorporates a statistical density-adaptive Pythagorean membership (SDAPM) scheme to dynamically adjust membership and non-membership based on class variance, local sparsity, and granular-ball compactness.

A. Fuzzy Membership

The membership degree of a fuzzy granular ball reflects its reliability and importance in learning. Two cases are considered for determining δ_{GB_i}. If prior membership information is available, the granular-ball membership is computed as the average of its samples. Since this is usually unavailable, a suitable membership function must be designed. We adopt a simple and effective distance-based strategy, where samples closer to the class center receive higher membership values, while distant samples receive lower values due to greater uncertainty.

Let x^+ and x^- denote the mean vectors of the positive and negative classes, respectively. The class radii are defined as:

$$r^+ = \max_{x_i, y_i=+1} |x_i - x^+|, \quad r^- = \max_{x_i, y_i=-1} |x_i - x^-|. \tag{2}$$

Based on these quantities, the fuzzy membership of a sample x_i is computed as:

$$\mu(x_i) = \begin{cases} 1 - \dfrac{|x_i - x^+|}{r^+ + \varepsilon}, & y_i = +1, \\ 1 - \dfrac{|x_i - x^-|}{r^- + \varepsilon}, & y_i = -1, \end{cases} \tag{3}$$

where $\varepsilon > 0$ is a small constant introduced to prevent zero membership values. Finally, the membership degree of a fuzzy granular-ball GB_i is obtained by aggregating the memberships of the samples contained within it, defined as:

$$\delta_{GB_i} = \frac{1}{|GB_i|} \sum_{x_j \in GB_i} \mu(x_j). \tag{4}$$

B. Statistical Density-Adaptive Pythagorean Membership (SDAPM)

The baseline membership assignment depends strongly on the maximum radius R_{max}, making it sensitive to outliers. To address this issue, we propose a scheme based on class-wise variance and local sparsity.

1) Statistical Membership Degree: Instead of a linear distance decay, we utilize a Gaussian Statistical Kernel. We define the class center C_j and the standard deviation of distances σ_j for each class $j \in \{+1, -1\}$. The membership degree of a granular-ball GB_i to its assigned class y_i is defined as:

$$\mu_{GBi} = \exp\left(-\frac{\|c_i - C_{y_i}\|^2}{2(\sigma_{y_i} \cdot \lambda_{y_i})^2}\right), \tag{5}$$

where $\lambda_{y_i} = \sqrt{\frac{m}{m_{y_i}}}$ represents the imbalance balancer.

2) Non-membership via Local Sparsity Index: When purity is unavailable, the non-membership degree is defined using a local sparsity index based on the proportion of nearby granular balls with different labels:

$$LSI_i = \frac{1}{k} \sum_{j \in KNN(GB_i)} \mathbb{I}(y_j \neq y_i). \tag{6}$$

The pythagorean non-membership degree ν_{GBi} is then derived such that it satisfies the constraint $\mu_{GBi}^2 + \nu_{GBi}^2 \leq 1$:

$$\nu_{GBi} = \sqrt{(1 - \mu_{GBi}^2) \cdot LSI_i}. \tag{7}$$

3) Ball Compactness and the Scoring Function: To differentiate stable core balls from uncertain boundary balls, we define ball compactness using the radius as a measure of certainty:

$$\omega_i = \exp\left(-\frac{r_i}{\bar{r} + \epsilon}\right), \tag{8}$$

where $\bar{r}$ is the average radius of all granular balls. The final scoring function s_{GBi} blends the membership degree with the pythagorean closeness index θ_{GBi}:

$$s_{GBi} = \omega_i \cdot \mu_{GBi} + (1 - \omega_i) \cdot \theta_{GBi}. \tag{9}$$

C. Fuzzy Granular Ball Random Vector Functional Link Network

Let $\mathcal{G} = \{(c_1, \delta_{GB_1}, y_1), (c_2, \delta_{GB_2}, y_2), \ldots, (c_k, \delta_{GB_k}, y_k)\}$, where c_i and δ_{GB_i} denote the center and membership degree of the granular ball GB_i, respectively, and y_i is its associated class label.

Let the hidden-layer matrix be denoted by $\mathcal{K}$, which is defined as:

$$\mathcal{K} = \phi(\mathcal{C}\mathcal{W} + b) \in \mathbb{R}^{k \times h}, \tag{10}$$

where ϕ is the activation function, $\mathcal{W} \in \mathbb{R}^{d \times h}$ is the randomly initialized weight matrix and $b \in \mathbb{R}^{1 \times h}$ denotes the bias vector.

The optimization problem of the proposed GBFRVFL model is given by:

$$\Theta_{\min} = \arg\min_{\Theta} \frac{\mathcal{D}}{2}\|S\eta\|^2 + \frac{1}{2}\|\Theta\|^2$$
$$\text{s.t. } \mathcal{H}\Theta - \mathcal{Y} = \eta, \tag{11}$$

where $\mathcal{H} = [h(c_1), h(c_2), \ldots, h(c_k)]^T \in \mathbb{R}^{k \times (d+h)}$, $h(c_i) = [c_i \;\; \varphi(c_i)]$, $S = diag(\delta_{GB_1}, \delta_{GB_2}, \ldots, \delta_{GB_k})$, and $\eta = [\eta_1, \eta_2, \ldots, \eta_k]^T$ is the error term corresponding to k granular balls. The Lagrangian corresponding to the problem (11) is given by

$$\mathcal{L}(\Theta, \eta, \lambda) = \frac{\mathcal{D}}{2}\|S\eta\|^2 + \frac{1}{2}\|\Theta\|^2 - \lambda^T(\mathcal{H}\Theta - \mathcal{Y} - \eta), \tag{12}$$

where λ is the Lagrangian multiplier. Using the Karush-Kuhn-Tucker (K.K.T.) conditions, we have

$$\frac{\partial \mathcal{L}}{\partial \Theta} = \Theta - \mathcal{H}^T\lambda, \tag{13}$$
$$\frac{\partial \mathcal{L}}{\partial \eta} = \mathcal{D}S^T(S\eta) + \lambda, \tag{14}$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = \mathcal{H}\Theta - \mathcal{Y} - \eta. \tag{15}$$

By putting Eq. (15) into Eq. (14), we obtain:

$$\lambda = -\mathcal{D}S^T S(\mathcal{H}\Theta - \mathcal{Y}). \tag{16}$$

By replacing λ with the value derived in Eq. (13), we get

$$\Theta = \mathcal{H}^T(-\mathcal{D}S^T S(\mathcal{H}\Theta - \mathcal{Y})),$$
$$\implies \Theta = \left(\frac{1}{\mathcal{D}}I + (S\mathcal{H})^T(S\mathcal{H})\right)^{-1}(S\mathcal{H})^T S\mathcal{Y}, \tag{17}$$

where I is the identity matrix of appropriate dimension. Next, by substituting the expressions from Eqs. (13)and (15) into Eq. (14), we obtain

$$\lambda = -\mathcal{D}S^T S(\mathcal{H}\mathcal{H}^T\lambda - \mathcal{Y}),$$
$$\implies \lambda + \mathcal{D}S^T S\mathcal{H}\mathcal{H}^T\lambda = \mathcal{D}S^T S\mathcal{Y},$$
$$\implies \lambda = \left(\frac{1}{\mathcal{D}}I + S^T S\mathcal{H}\mathcal{H}^T\right)^{-1} S^T S\mathcal{Y}. \tag{18}$$

Put the value of λ in (13), we get

$$\Theta = \mathcal{H}^T\left(\frac{1}{\mathcal{D}}I + S^T S\mathcal{H}\mathcal{H}^T\right)^{-1} S^T S\mathcal{Y}. \tag{19}$$

Two equations can compute Θ, requiring matrix inversion. If $(d + h) \leq k$, Eq. (17) is used; if $(d + h) > k$, Eq. (19) is used. This allows inversion in feature or sample space, giving the optimal solution of Eq. (11).

$$\Theta = \begin{cases} \left(\frac{1}{\mathcal{D}}I + (S\mathcal{H})^T(S\mathcal{H})\right)^{-1}(S\mathcal{H})^T S\mathcal{Y}, & \text{if } (d+h) \leq k, \\ \mathcal{H}^T\left(\frac{1}{\mathcal{D}}I + S^T S\mathcal{H}\mathcal{H}^T\right)^{-1} S^T S\mathcal{Y}, & \text{if } k < (d+h). \end{cases} \tag{20}$$

The algorithms of the proposed GBFRVFL model are given in Algorithm 1.

Algorithm 1 Algorithm of the proposed GBFRVFL model.

Input: Training set $\mathcal{M}$, and the purity threshold ρ.

Output: The output weight matrix of the GBFRVFL model.

1: Initialize the full dataset as a granular ball $GB = \mathcal{M}$ and set the granular-ball set $\mathcal{G}$ to empty.
2: $\Psi = \{GB\}$.
3: **for** $i = 1 : |\Psi|$ **do**
4:     **if** $\mathrm{pur}(GB_i) < \rho$ **then**
5:         Split GB_i into two subsets, GB_{i1} and GB_{i2}, using 2-means clustering.
6:         $\Psi \leftarrow \Psi \cup \{GB_{i1}, GB_{i2}\}$
7:     **else if** $pur(GB_i) \geq \rho$ **then**
8:         Compute the centroid of GB_i as $c_i = \frac{1}{n_i}\sum_{i=1}^{n_i} x_i$, where n_i is the number of samples in GB_i.
9:         Assign GB_i the label y_i of its most frequent class.
10:         Put $GB_i = \{(c_i, y_i)\}$ in $\mathcal{G}$.
11:     **end if**
12: **end for**
13: **if** $\Psi \neq \emptyset$ **then**
14:     Return to Step 3 to continue hierarchical decomposition.
15: **else**
16:     Compute the membership degree of GB_i as δ_{GB_i}.
17: **end if**
18: Form the set $\mathcal{G} = GB_{i}{}_{i=1}^{k} = (c_i, \delta_{GB_i}, y_i)_{i=1}^{k}$, with each ball defined by its center, membership degree, and label.
19: Compute the hidden-layer features using (10).
20: Calculate the enhanced features matrix using $\mathcal{H} = [\mathcal{C},\ \mathcal{K}]$.
21: Calculate the output layer weights using (20).

IV. Numerical Experiments and Results

To evaluate the GBFRVFL framework, two membership strategies are integrated into the GBRVFL architecture: **F-GBRVFL** (fuzzy membership) and **SDAP-GBRVFL** (SDAP membership). Both models are tested on 37 UCI and KEEL benchmark datasets and compared with RVFL [6], ELM [7], GB-RVFL and GE-GB-RVFL [24], and CRVFL and ACRVFL [25]. Experiments with added label noise are provided in Section S.I, and sensitivity analyses are presented in Section S.II of the supplementary material.

A. Experimental Setup

Experiments are run on a Windows 11 workstation with an Intel Xeon Gold 6226R (2.90 GHz) and 256GB RAM using Python 3.11. Datasets are split 70:30 for training and testing, with hyperparameters tuned via grid search and five-fold cross-validation, where the regularization parameters are explored within the set $\mathcal{D} = \{10^{-5}, 10^{-4}, \ldots, 10^{5}\}$. Hidden nodes h_l range from 3 to 203, and nine activation functions are tested: SELU, ReLU, Sigmoid, Sine, Hardlim, Tribas, Radbas, Sign, and Leaky ReLU.

B. Evaluation on UCI and KEEL Datasets

The classification effectiveness of the proposed F-GBRVFL and SDAP-GBRVFL models, along with the corresponding results of the baseline models, is evaluated in terms of classification accuracy (Acc), and the comparative outcomes are given in Table I. The proposed F-GBRVFL and SDAP-GBRVFL models achieve an average Acc of 82.29% and 84.46, whereas the baseline models RVFL, ELM, GB-RVFL, GE-GB-RVFL, CRVFL, and ACRVFL attain an average Acc of 81.55%, 80.81%, 79.62%, 78.17%, 75.86%, and 76.62%, respectively. In terms of average Acc, the proposed F-GBRVFL and SDAP-GBRVFL models consistently outperform the competing models, demonstrating their robust and dependable predictive performance. However, while average Acc offers an overall performance, it may not fully reflect model behavior, since superior results on some datasets can be offset by relatively weaker performance on others. To address this limitation and provide a more rigorous assessment of whether the observed performance differences are statistically meaningful, a series of nonparametric statistical tests, as recommended by Demšar [26], are employed. These statistical methods are suitable for comparing multiple learning algorithms across diverse datasets when parametric test assumptions are not met. Accordingly, this study uses nonparametric techniques, including rank-based methods, the Friedman test, and Nemenyi post hoc analysis, to ensure objective performance comparisons. In rank-based evaluation, models are ranked within each dataset and aggregated across datasets, with better-performing models receiving lower ranks and poorer-performing models receiving higher ranks. For a comparison involving ξ models evaluated on N datasets, let $\mathcal{R}_i^j$ denote the rank assigned to the j^{th} model on the i^{th} dataset. The overall average rank of the j^{th} model is then calculated as $\mathcal{R}^j = \frac{1}{n}\sum_{i=1}^{N}\mathcal{R}_i^j$. The average rank of our proposed F-GBRVFL and SDAP-GBRVFL models, along with the baseline RVFL, ELM, GB-RVFL, GE-GB-RVFL, CRVFL, and ACRVFL models, are 3.2, 2.07, 4.51, 5.22, 4.86, 5.16, and 5.26, respectively. The F-GBRVFL and SDAP-GBRVFL models attain the smallest average rank across all evaluated models. As lower rank values indicate better performance, this result highlights the proposed F-GBRVFL and SDAP-GBRVFL as the most effective approaches among the models under comparison. The Friedman test [27] is a nonparametric statistical technique employed to examine whether meaningful performance differences exist among multiple models by comparing their average ranks over several datasets. The null hypothesis assumes no significant differences in model performance. The Friedman test statistic, denoted by χ_F^2, is assumed to follow a chi-square distribution with $(\xi - 1)$ degrees of freedom and is calculated as $\chi_F^2 = \frac{12N}{\xi(\xi+1)}\left[\sum_j \mathcal{R}_j^2 - \frac{\xi(\xi+1)^2}{4}\right]$. In addition, the corresponding F_F statistic is given by $F_F = \frac{(N-1)\chi_F^2}{N(\xi-1)-\chi_F^2}$,

TABLE I: Performance comparison of the proposed F-GBRVFL and SDAP-GBRVFL models along with the baseline models for UCI and KEEL datasets.

Dataset	RVFL [6]	ELM [7]	GB-RVFL [24]	GE-GB-RVFL [24]	CRVFL [25]	ACRVFL [25]	F-GBRVFL†	SDAP-GBRVFL†
bank	89.17	87.31	88.43	88.95	88.28	88.8	89.09	89.24
blood	76.44	72.14	76	77.33	75.56	77.33	79.11	80.33
breast_cancer	72.09	74.42	77.91	65.12	79.07	74.42	74.42	74.93
breast_cancer_wisc_prog	75	73.33	73.33	73.33	66.67	66.67	76.67	76.67
bupa or liver-disorders.csv	65.38	65.38	54.81	56.73	94.95	77.24	62.5	67.54
checkerboard_Data.csv	85.94	85.98	87.02	87.02	62.48	71.5	87.02	87.5
chess_krvkp	90.41	90.2	91.45	89.16	93.95	94.06	93.64	94.58
cleve.csv	81.11	80	75.56	82.22	73.79	79.79	81.11	81.11
conn_bench_sonar_mines_rocks	74.6	71.43	74.6	68.25	65.08	66.67	73.9	74.43
credit_approval	84.62	82.3	77.88	80.77	83.65	84.13	84.65	84.62
crossplane150.csv	81.11	81.11	86.67	73.33	84.68	73.79	83.33	88.89
cylinder_bands	72.73	70.67	59.74	64.29	70.78	66.23	71.43	72.88
ecoli-0-1-4-6_vs_5.csv	98.81	98.81	98.81	98.81	87.14	86.74	98.81	99.62
ecoli-0-1-4-7_vs_5-6.csv	90	96	94	94	72.9	70.66	70	96
fertility	90	80	86.67	90	90	90	90	90
haber.csv	76.09	78.26	78.26	77.17	75.46	85.85	79.35	80.17
haberman.csv	76.09	76.09	78.26	77.17	71.58	83.33	83.35	84.17
haberman_survival	76.09	78.26	78.26	77.17	82.61	82.61	82.35	83.26
heart_hungarian	72.78	72.65	74.16	74.16	75.28	76.4	75.28	80.9
hepatitis	72.34	70.11	72.34	72.34	76.6	82.98	72.34	76.6
hill_valley	68.41	67.31	59.07	60.71	53.3	54.12	66.48	68.13
horse_colic	83.78	80.18	72.97	76.58	76.58	77.48	83.78	86.49
ionosphere	82.79	82.45	83.96	83.02	85.85	83.96	84.91	83.96
monks_3	90.41	90.41	85.63	91.02	74.85	85.03	92.22	90.42
new-thyroid1.csv	86	86	90.77	96.92	73.15	67.62	89.23	100
oocytes_merluccius_nucleus_4d	80.71	79.74	80.78	78.5	67.43	77.2	81.76	81.11
oocytes_trisopterus_nucleus_2f	80.12	80.94	75.55	80.29	69.71	71.9	83.21	76.28
statlog_australian_credit	68.75	68.75	62.02	62.02	70.19	70.19	70.67	71.63
statlog_german_credit	70.33	70.67	70	75.33	69	66.33	77	75.33
vehicle2.csv	90.85	90.03	91.73	92.52	90.36	85.9	94.88	97.24
vertebral_column_2clases	81.4	72.25	74.19	74.19	66.67	77.42	88.17	87.1
votes.csv	90.47	90.47	92.37	87.02	67.22	67.78	95.42	97.71
vowel.csv	99.33	98.99	85.19	63.97	70.71	67.03	94.95	92.59
wpbc.csv	69.49	70.97	76.27	69.49	75.69	84.96	64.41	72.88
yeast-0-2-5-6_vs_3-7-8-9.csv	89.71	89.71	93.38	90.73	77.16	67.02	93.71	92.72
yeast-0-2-5-7-9_vs_3-6-8.csv	95.35	95.68	98.01	97.35	78.44	70.71	88.74	97.02
yeast-0-3-5-9_vs_7-8.csv	88.82	90.79	69.74	45.39	70.01	81.01	86.84	90.79
Average Acc	81.55	80.81	79.62	78.17	75.86	76.62	<u>82.29</u>	**84.46**
Average Rank	4.51	5.22	4.86	5.16	5.72	5.26	3.2	2.07

The proposed model is denoted by †.
The top and second-best models in terms of ACC are denoted by boldface and underline, respectively.

which follows an F-distribution with $(\xi-1)$ and $(N-1)(\xi-1)$ degrees of freedom. For $N = 37$ datasets and $\xi = 8$ competing models, the Friedman test produces a test statistic of $\chi_F^2 = 66.26$, leading to an F_F value of 12.38. At the 5% significance level, the critical value of the F-distribution with $(7, 252)$ degrees of freedom is 2.05. Since F_F exceeds the threshold, the null hypothesis is rejected, indicating significant performance differences. Next, the Nemenyi post hoc test is employed to analyze the pairwise differences in performance between the models. The critical difference (C.D.) is determined using: C.D. $= q_\alpha\sqrt{\frac{\xi(\xi+1)}{6N}}$, where q_α represents the critical value from the two-tailed Nemenyi distribution. At a 5% significance level, $q_\alpha = 3.031$, which yields a corresponding C.D. of 1.72. The differences in average ranks between the proposed F-GBRVFL and SDAP-GBRVFL models and the baseline RVFL, ELM, GB-RVFL, GE-GB-RVFL, CRVFL, and ACRVFL models are $(1.31, 2.44)$, $(2.02, 3.15)$, $(1.66, 2.79)$, $(1.96, 3.09)$, $(2.52, 3.65)$, and $(2.06, 3.19)$, respectively. This difference indicates that F-GBRVFL outperforms all baseline models except RVFL and GB-RVFL, showing consistently strong performance, while SDAP-GBRVFL achieves statistically significant improvements over the baselines.

V. Conclusion

In this paper, we introduced a novel granular-ball-based randomized learning framework, termed GBFRVFL, which leverages the coarse-grained abstraction of granular balls to enhance robustness and generalization in the presence of noise, outliers, and class imbalance. Two variants of the proposed model are developed to address uncertainty and improve classification performance. The first, F-GBRVFL, integrates a fuzzy membership scheme to assign degrees of reliability to each granular ball, effectively mitigating the influence of noisy or ambiguous samples. The second, SDAP-GBRVFL, incorporates a novel statistical density-adaptive pythagorean membership (SDAPM) scheme that dynamically adjusts membership and non-membership values based on class variance, local sparsity, and granular-ball compactness, ensuring effective handling of uncertain granular-ball distributions and class imbalance. Extensive experiments on UCI and KEEL benchmark datasets, including scenarios with varying levels of label noise, demonstrate that both F-GBRVFL and SDAP-GBRVFL consistently outperform existing randomized learning and granular ball-based models. SDAP-GBRVFL shows strong robustness to noise and better preserves intrinsic data structure while retaining RVFL's computational efficiency. Statistical analyses confirm significant improvements, validating the proposed membership schemes. This work focuses on shallow RVFL networks, with future extensions to deep and ensemble models.

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