Fully distributed adaptive pinning control for fractional-order multiplex networks

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***Abstract*—This work studies distributed adaptive strategies for achieving intra-layer synchronization in fractional-order multiplex networks via pinning control.Two distributed adaptive strategies are introduced: a node-based strategy, where the coupling strength of each node is dynamically updated using local state information from both the node and its neighbors; and an edge-based strategy, in which the weight of each intra-layer edge is updated according to the relative states between the two linked nodes. Additionally, the pinning gains applied to a subset of nodes are adaptively adjusted based on the discrepancy between each pinned node and its reference state. Rigorous theoretical analysis and numerical simulations are provided to demonstrate the effectiveness of the proposed algorithms.**

***Index Terms*—Fractional-order multiplex networks, adaptive algorithms, intra-layer synchronization**

I. Introduction

Control and synchronization of linearly coupled networks have attracted sustained attention in the study of complex systems. In many real-world scenarios, networked nodes interact through diverse and heterogeneous mechanisms, as commonly observed in social systems [1], biological networks [2], and transportation infrastructures [3]. To systematically model these multi-modal interactions, the framework of multiplex networks was formally established [4]. Such networks comprise multiple layers that share the same node set, with each layer capturing a distinct interaction type among the nodes. In recent years, synchronization dynamics within multiplex networks have attracted significant scholarly attention [5][10].

Adaptive control provides an effective framework for addressing control and synchronization problems in complex networks. Existing studies on adaptive control can be broadly classified into two categories: node-based and edge-based strategies. In node-based adaptive schemes, the control parameters of each node are adjusted using relative state information from the node itself and its neighboring nodes [11], [12]. By contrast, edge-based adaptive schemes update the coupling weights associated with network edges according to the relative information between the two incident nodes [13], [14], [15]. Notably, most of the aforementioned results focus on single-layer network structures. More recently, [16] extended both node-based and edge-based adaptive control strategies to investigate intra-layer synchronization in integer-order multiplex networks.

In recent years, fractional-order calculus has attracted growing interest due to its superior capability in characterizing infinite memory and hereditary effects in system modeling, particularly in bioengineering, neural networks, and fluid mechanics [17], [18]. Fractional-order complex networks can be regarded as a natural extension of conventional integer-order network models, providing greater flexibility in describing system dynamics. Synchronization control of fractional-order multiplex networks has therefore received increasing attention. For instance, [19] investigated synchronization of fractional-order multiplex networks with time delays using adaptive and impulsive control strategies, while [20] achieved synchronization with output couplings via adaptive quantized controllers. Notably, these studies require controllers to be applied to all nodes.

In contrast, pinning control only acts on a fraction of nodes and can significantly reduce control costs in large-scale networks. Along this line, [23] designed a quantized pinning scheme for output synchronization of fractional-order multi-layer networks under stochastic cyber attacks. In that work, only the pinning strengths of selected nodes are adaptively adjusted, and synchronization conditions are derived in terms of a linear matrix inequality involving the network topology, uncoupled node dynamics, and intra-layer coupling strengths.

Motivated by the above discussions, this paper proposes a node-based adaptive algorithm and an edge-based adaptive algorithm for intra-layer synchronization of fractional-order multiplex networks with pinning control. By employing the Lyapunov function method, it is theoretically shown that the proposed algorithms guarantee pinning synchronization of fractional-order multiplex networks provided that the intra-layer network topologies are strongly connected.

II. Preliminaries and model description

A. Preliminaries

This subsection introduces the necessary notations and lemmas used throughout the paper. For a vector x, $\|x\|$ denotes its Euclidean norm. For a square matrix A, A_{sym} denotes its symmetric part, defined as $A_{sym} = (A + A^\top)/2$.

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Some preliminaries on fractional-order derivatives and integrals are presented below.

Definition 1. *The fractional-order integral of order $0 < \alpha \leq 1$ for an integrable function $h(t) : [t_0, \infty) \to \mathbb{R}$ is defined by*

$$ {}_{t_0}I_t^{\alpha} h(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{h(s)}{(t-s)^{(1-\alpha)}} ds, \tag{1}$$

where $\Gamma(\alpha) = \int_0^{+\infty} e^{-t} t^{\alpha-1} dt$.

Definition 2. *The Caputo fractional-order derivative of order $0 < \alpha \leq 1$ for a differentiable function $h(t) : [t_0, \infty) \to \mathbb{R}$ is defined by*

$$ {}_{t_0}^{C}D_t^{\alpha} h(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{h'(s)}{(t-s)^{\alpha}} ds. \tag{2}$$

When $\alpha = 1$, the Caputo derivative reduces to the classical first order derivative, that is ${}_{t_0}^{C}D_t^{1} h(t) = h'(t)$.

Lemma 1. *(Lemma 1 in [20]) Let $h(t) : [t_0, \infty) \to \mathbb{R}$ be a continuously differentiable function. Then*

$$ {}_{t_0}I_t^{\alpha} {}_{t_0}^{C}D_t^{\alpha} h(t) = h(t) - h(t_0) \tag{3}$$

where $\alpha \in (0, 1)$.

Lemma 2. *(Lemma 2 in [20]) Assume that $H(t) : [t_0, \infty) \to \mathbb{R}^n$ is continuously differentiable. Then*

$$ {}_{t_0}^{C}D_t^{\alpha} H(t)^{\top} H(t) \leq 2H(t)^{\top} {}_{t_0}^{C}D_t^{\alpha} H(t) \tag{4}$$

B. Model description

This article investigates node-based and edge-based fully distributed adaptive algorithms for intra-layer synchronization in fractional-order multiplex networks. To drive the dynamics in different layers toward prescribed target trajectories, feedback controllers with adaptive gains are applied to a small subset of nodes in each layer.

We consider the following fractional-order multiplex network with adaptive coupling parameters and pinning controllers:

$$\begin{aligned} {}_{0}^{C}D_t^{\alpha} x_i^k(t) &= f_k(x_i^k(t)) + c_i^k(t) \sum_{j=1}^{N} w_{ij}^k(t) \left[x_j^k(t) - x_i^k(t) \right] \\ &+ p \sum_{l=1}^{M} b_{kl} \left[x_i^l(t) - x_i^k(t) \right] + c_{i0}^k(t) \left[s_k(t) - x_i^k(t) \right], \end{aligned} \tag{5}$$

where $\alpha \in (0, 1]$. The scalar $c_i^k(t)$ denotes the coupling strength of the ith node in the kth layer, and $W^k(t) = [w_{ij}^k(t)] \in \mathbb{R}^{N \times N}$, with $w_{ii}^k = -\sum_{j \neq i} w_{ij}^k$, represents the intra-layer coupling matrix of the kth layer. The matrix $B = [b_{kl}] \in \mathbb{R}^{M \times M}$, with $b_{kk} = -\sum_{l \neq k} b_{kl}$, denotes the inter-layer coupling matrix. Moreover, $c_{i0}^k(t)$ is the pinning strength, and $s_k(t)$ represents the target trajectory of the kth layer. For any $i \notin \mathcal{D}_k$, it holds that $c_{i0}^k(t) = 0$ for all t.

To ensure intra-layer synchronization of the multiplex network under pinning control, the target trajectories must be invariant under system (5). Consequently, the target trajectories $s_k(t)$, $k = 1, \ldots, M$, should satisfy

$$ {}_{0}^{C}D_t^{\alpha} s_k(t) = f_k(s_k(t)) + p \sum_{l=1}^{M} b_{kl} \left[s_l(t) - s_k(t) \right]. $$

Let $G^k = (\mathcal{V}, \mathcal{E}^k, A^k)$ denote the initial intra-layer graph topology of the kth layer, where $\mathcal{V}$ is the common node set of the multiplex network, $\mathcal{E}^k$ denotes the edge set of the kth layer, and $A^k = [a_{ij}^k]$, with $a_{ii}^k = -\sum_{j \neq i} a_{ij}^k$, is the corresponding coupling matrix. Let $\mathcal{N}_i^k$ denote the neighborhood of node i in the kth layer, and let $\mathcal{D}_k$ denote the set of pinned nodes in the kth layer. Throughout this paper, the following assumption is imposed.

Assumption 1. *All intra-layer graph topologies G^k, $k = 1, \cdots, M$ are strongly connected.*

For any k, let $D^k = \operatorname{diag}(d_1^k, \ldots, d_N^k)$ denote the pinning matrix of the kth layer, where $d_i^k = 1$ if $i \in \mathcal{D}_k$ and $d_i^k = 0$ otherwise. For a given constant $\epsilon > 0$, define $\tilde{A}^k = A^k - \epsilon D^k$. The following lemma characterizes an important property of $\tilde{A}^k$.

Lemma 3. *(Lemma 1 in [16]) Suppose that Assumption 1 holds. For any $k = 1, \ldots, M$, the matrix A^k admits a positive normalized left eigenvector θ^k associated with the zero eigenvalue, satisfying $\sum_{i=1}^{N} \theta_i^k = 1$. Let $\Theta^k = \operatorname{diag}(\theta^k)$. Then the symmetric part $\{\Theta^k \tilde{A}^k\}_{\mathrm{sym}}$ is negative definite.*

Notation 1. *For any k, the eigenvalues of $\{\Theta^k \tilde{A}^k\}_{sym}$ are denoted by $0 > \tilde{\lambda}_1^k \geq \tilde{\lambda}_2^k \geq \cdots \geq \tilde{\lambda}_N^k$.*

Throughout this paper, the node dynamics are assumed to satisfy the following condition.

Assumption 2. *For any k, $f_k(x)$ satisfies the QUAD condition, that is, there exists a constant $q_k \in \mathbb{R}$ such that*

$$ (x-y)^{\top} \left[f_k(x) - f_k(y) \right] \leq q_k (x-y)^{\top} (x-y) \tag{6}$$

holds for all $x, y \in \mathbb{R}^n$.

III. Fully distributed adaptive intra-layer synchronization algorithms with pinning control

This section investigates fully distributed adaptive algorithms for intra-layer synchronization in multiplex networks with pinning control. Two classes of adaptive strategies are considered: node-based and edge-based algorithms.

We first present the node-based adaptive algorithm, in which each node updates its coupling strength using relative state information from itself and its neighbors.

A. Node-Based Algorithm

For $i = 1, \cdots, N$ and $k = 1, \cdots, M$, the coupling strength $c_i^k(t)$ is adaptively updated according to

$$ {}_{0}^{C}D_t^{\alpha} c_i^k(t) = \delta_i^k \sum_{j \in \mathcal{N}_i^k} a_{ij}^k \left[x_i^k(t) - x_j^k(t) \right]^{\top} \left[x_i^k(t) - x_j^k(t) \right], \tag{7}$$

where $\delta_i^k > 0$ is the adaptive gain. For each pinned node $i \in \mathcal{D}_k$, the pinning strength $c_{i0}^k(t)$ is updated as

$$ {}_0^C D_t^\alpha c_{i0}^k(t) = \delta_{i0}^k \big[x_i^k(t) - s_k(t)\big]^\top \big[x_i^k(t) - s_k(t)\big], \tag{8}$$

where $\delta_{i0}^k > 0$. In this node-based strategy, the coupling weights of all edges remain constant, i.e., $w_{ij}^k(t) = a_{ij}^k$.

The following theorem establishes the main result of this paper, showing that the multiplex network (5) can achieve intra-layer synchronization under adaptive intra-layer coupling and pinning control.

Theorem 1. *Suppose that Assumptions 1 and 2 hold. Then the multiplex network* (5) *with the updating laws* (7) *and* (8) *achieves intra-layer synchronization.*

Proof: For each $i = 1, \ldots, N$ and $k = 1, \ldots, M$, define the synchronization error

$$e_i^k(t) = x_i^k(t) - s_k(t). \tag{9}$$

Consider the following Lyapunov candidate function:

$$\begin{aligned} V(t) =& \frac{1}{2} \sum_{i,k} \theta_i^k e_i^k(t)^\top e_i^k(t) + \frac{1}{4\delta_i^k} \sum_{i,k} \big[c_i^k(t) - \beta\theta_i^k\big]^2 \\ &+ \frac{1}{2\delta_{i0}^k} \sum_{k, i\in\mathcal{D}_k} \big[c_{i0}^k(t) - \beta\theta_i^k \epsilon\big]^2, \end{aligned} \tag{10}$$

where $\beta > 0$ is a constant to be determined.

Taking the Caputo fractional derivative of $V(t)$ along system (5) and the updating laws (7)(8) yields

$$\begin{aligned} {}_0^C D_t^\alpha V(t) =& \sum_{i,k} \theta_i^k {e_i^k}^\top f_k(x_i^k) - \sum_{i,k} \theta_i^k {e_i^k}^\top f_k(s_k) \\ &+ \sum_{i,j,k} \theta_i^k {e_i^k}^\top c_i^k a_{ij}^k e_j^k + p \sum_{i,k,l} \theta_i^k {e_i^k}^\top b_{kl} e_i^l \\ &+ \frac{1}{2} \sum_{i,j,k} \big[c_i^k - \beta\theta_i^k\big] a_{ij}^k \|e_i^k - e_j^k\|^2 \\ &+ \sum_{k, i\in\mathcal{D}_k} \big[c_{i0}^k - \beta\theta_i^k\epsilon\big] \|e_i^k\|^2. \end{aligned} \tag{11}$$

By Assumption 2, one has

$$\sum_{i,k} \theta_i^k {e_i^k}^\top \big[f_k(x_i^k) - f_k(s_k)\big] \le \sum_{i,k} q_k \theta_i^k \|e_i^k\|^2. \tag{12}$$

From Lemma 3 and Notation 1, it follows that

$$\beta \sum_{i,j,k} \theta_i^k a_{ij}^k {e_i^k}^\top e_j^k \le \beta \sum_{i,k} \tilde{\lambda}_1^k \|e_i^k\|^2. \tag{13}$$

Define $\hat{c}(t) = \max\{\max_{i,k} c_i^k(t), \max_{i,k} c_{i0}^k(t)\}$ and $\bar{a} = \max_{i,j,k} a_{ij}^k$. Using the inequality $2u^\top v \le u^\top u + v^\top v$, one obtains

$$\sum_{i,j,k} \theta_i^k {e_i^k}^\top c_i^k a_{ij}^k e_j^k \le N\bar{a}\,\hat{c}(t) \sum_{i,k} \|e_i^k\|^2, \tag{14}$$

$$\sum_{i,j,k} c_i^k a_{ij}^k \|e_i^k - e_j^k\|^2 \le 2N\bar{a}\,\hat{c}(t) \sum_{i,k} \|e_i^k\|^2, \tag{15}$$

$$\sum_{k, i\in\mathcal{D}_k} c_{i0}^k \|e_i^k\|^2 \le \hat{c}(t) \sum_{i,k} \|e_i^k\|^2. \tag{16}$$

Let $\bar{b} = \max_{k, l\neq k} |b_{kl}|$. Then

$$p \sum_{i,k,l} \theta_i^k {e_i^k}^\top b_{kl} e_i^l \le pM\bar{b} \sum_{i,k} \|e_i^k\|^2. \tag{17}$$

Combining the above inequalities yields

$$ {}_0^C D_t^\alpha V(t) \le \left[\eta + \beta \max_k \tilde{\lambda}_1^k + (3N\bar{a}+1)\hat{c}(t)\right] V_1(t), \tag{18}$$

where $V_1(t) = \sum_{i,k} \|e_i^k(t)\|^2$ and $\eta = pM\bar{b} + \max_k q_k \bar{\theta}^k$.

Let

$$V_1(t) = \sum_{i,k} e_i^k(t)^\top e_i^k(t). \tag{19}$$

By $x_i^k(t) - x_j^k(t) = e_i^k(t) - e_j^k(t)$ and the dynamics of ${}_0^C D_t^\alpha c_i^k(t)$ in (7),

$$\begin{aligned} {}_0^C D_t^\alpha c_i^k(t) \le& 2\delta_i^k \sum_{j\in\mathcal{N}_i^k} a_{ij}^k \left[e_i^k(t)^\top e_i^k(t) + e_j^k(t)^\top e_j^k(t)\right] \\ \le& 2N\bar{a}\delta_i^k V_1(t), \end{aligned} \tag{20}$$

Combing with ${}_0^C D_t^\alpha c_{i0}^k(t)$ in (8) and Lemma 1, one has that

$$\hat{c}(t) \le \hat{c}(0) + 2N\bar{a}\bar{\delta} \cdot {}_{t_0}I_t^\alpha V_1(t), \tag{21}$$

where $\bar{\delta} = \max_{i,k}\{\delta_i^k, \delta_{i0}^k\}$. Then by (18) and (21), we have

$$\begin{aligned} &{}_{t_0}^C D_t^\alpha V(t) \\ &\le \left[\tilde{\eta} + \beta \max_k \tilde{\lambda}_1^k + (6N^2\bar{a}^2 + 2N\bar{a})\bar{\delta} \cdot {}_0I_t^\alpha V_1(t)\right] V_1(t), \end{aligned} \tag{22}$$

where $\bar{\theta} = \max_{j,k} \theta_j^k$, $\bar{\theta}^k = \max_j \theta_j^k$, and $\tilde{\eta} = \eta + (3N\bar{a}+1)\,\hat{c}(0)$.

Since $V_1(t) \ge 0$ holds for any $t \ge 0$, the integral${}_0I_\tau^\alpha V_1(t)$ is nondecreasing with respect to τ. Therefore, for any given $T > 0$, if

$$\beta \ge \frac{1}{-\max_k \tilde{\lambda}_1^k} \left[\tilde{\eta} + (6N^2\bar{a}^2 + 2N\bar{a})\bar{\delta} \cdot {}_0I_T^\alpha V_1(t) + 1\right],$$

then for all $t \in [0, T]$,

$$\tilde{\eta} + \beta \max_k \tilde{\lambda}_1^k + (6N^2\bar{a}^2 + 2N\bar{a})\bar{\delta} \cdot {}_0I_t^\alpha V_1(t) \le -1.$$

Consequently, ${}_0^C D_t^\alpha V(t) \le -V_1(t)$ holds for all $t \in [0, T]$. From Lemma 1, one obtains that

$${}_0I_T^\alpha V_1(t) \le V(0) - V(T) \le V(0).$$

Due to the arbitrariness of T, it follows that if

$$\beta \ge \frac{1}{-\max_k \tilde{\lambda}_1^k} \left[\tilde{\eta} + (6N^2\bar{a}^2 + 2N\bar{a})\bar{\delta} \cdot V(0) + 1\right],$$

then for any $t > 0$, the following inequality holds:

$${}_0^C D_t^\alpha V(t) \le -V_1(t), \tag{23}$$

Note that $V_1(t) \le V(t)/\underline{\theta}$ holds for any $t > 0$, where $\underline{\theta} = \min_{j,k} \theta_j^k$. Then from Lemma 1 and inequality (23), one obtains that

$$V_1(t) \le \frac{1}{\underline{\theta}} \left[V(t) + {}_0I_t^\alpha V_1(t)\right] \le \frac{1}{\underline{\theta}} V(0) \tag{24}$$

which implies that $V_1(t)$ and ${}_0I_t^\alpha V_1(t)$ are both bounded for $t \geq 0$. Then we can get that $\lim_{t\to\infty} V_1(t) = \lim_{t\to\infty} \sum_{i,k} e_i^k(t)^\top e_i^k(t) = 0$. This completes the proof. ■

Remark 1. *From (20) and (24), we obtain*

$$\begin{aligned} c_i^k(t) - c_i^k(0) &= {}_0I_t^\alpha {}_0^C D_t^\alpha c_i^k(t) \leq 2N\bar{a}\bar{\delta} \cdot {}_0I_t^\alpha V_1(t) \\ &< 2N\bar{a}\bar{\delta}V(0)/\underline{\theta}, \end{aligned}$$

which implies that $c_i^k(t)$, for all $i = 1, \cdots, N$, $k = 1, \cdots, M$, are always bounded. Similarly, one can prove that c_{i0}^k are bounded for all $i \in \mathcal{D}_k$, $k = 1, \cdots, M$.

Remark 2. *In [20], a quantized adaptive controller is developed to realize synchronization of fractional-order multiplex networks. With the help of adaptive controllers on all nodes, the dynamics of each layer can synchronize to the target model. Compared with [20], the pinning scheme in this paper only needs to adaptively control a fraction of nodes.*

B. Edge-based algorithm

This subsection discusses the edge-based adaptive algorithm, in which each edge updates its coupling weight using relative state information from the two incident nodes.

The coupling weight $w_{ij}^k(t)$ in (5) is adaptively updated according to

$$ {}_0^C D_t^\alpha w_{ij}^k(t) = \sigma_{ij}^k [x_i^k(t) - x_j^k(t)]^\top [x_i^k(t) - x_j^k(t)],\ j \in \mathcal{N}_i^k \tag{25}$$

where $\sigma_{ij}^k > 0$ is the adaptive gain. For $j \notin \mathcal{N}_i^k$, one has ${}_0^C D_t^\alpha w_{ij}^k(t) = 0$, implying that $w_{ij}^k(t) = a_{ij}^k = 0$ holds for all time t.

In this edge-based strategy, the node coupling strengths remain constant, i.e., $c_i^k(t) = c_i^k$. The pinning strength c_{i0}^k are updated according to (8).

Assume that for all i, k, t,

$${}_0^C D_t^\alpha w_{ii}^k(t) = -\sum_{j\neq i} {}_0^C D_t^\alpha w_{ij}^k(t).$$

Then, for any $t > 0$,

$$\sum_j {}_0^C D_t^\alpha w_{ij}^k(t) = 0 \text{ and } \sum_j w_{ij}^k(t) = 0$$

which follows from the initial condition $\sum_j w_{ij}^k(0) = \sum_j a_{ij}^k = 0$.

The following theorem establishes intra-layer synchronization under the edge-based adaptive strategy.

Theorem 2. *Suppose Assumptions 1 and 2 hold. Then the multiplex network (5) with updating laws (25) and (8) achieves intra-layer synchronization.*

Proof: Consider the Lyapunov function

$$\begin{aligned} \hat{V}(t) =& \frac{1}{2}\sum_{i,k} \theta_i^k e_i^k(t)^\top e_i^k(t) + \frac{1}{4\sigma_{ij}^k}\sum_{i,j,k} [w_{ij}^k(t) - \gamma\theta_i^k a_{ij}^k]^2 \\ &+ \frac{1}{2\delta_{i0}^k}\sum_{k,i\in\mathcal{D}_k} [c_{i0}^k(t) - \gamma\theta_i^k \epsilon]^2, \end{aligned} \tag{26}$$

where $\gamma > 0$ is a constant to be determined.

Let

$$\hat{w}(t) = \max_{i,j,k}\{w_{ij}^k(t), c_{i0}^k(t)\}.$$

Using arguments similar to those in the proof of Theorem 1, one can show that

$$\hat{w}(t) \leq \hat{w}(0) + 2\bar{\sigma} \cdot {}_0I_t^\alpha V_1(t),$$

where $\hat{\sigma} = \max_{i,j,k}\{\sigma_{ij}^k, \delta_{i0}^k\}$ and $V_1(t)$ is defined in (19).

Following steps analogous to those in the proof of Theorem 1, the Caputo derivative of $\hat{V}(t)$ along (5) and (25) satisfies

$$\begin{aligned} {}_0^C D_t^\alpha \hat{V}(t) \leq& \sum_{i,k} q_k\theta_i^k e_i^k(t)^\top e_i^k(t) + \gamma\sum_{i,j,k} \theta_i^k \tilde{a}_{ij}^k e_i^k(t)^\top e_j^k(t) \\ &+ pM\bar{b}\sum_{i,k} e_i^k(t)^\top e_i^k(t) \\ &+ (\bar{c}N + 2N + 1)\hat{w}(t)\sum_{i,k} e_i^k(t)^\top e_i^k(t) \end{aligned}$$

where $\bar{c} = \max_{i,k} c_i^k$.

The remainder of the proof follows directly from Theorem 1 and is omitted for brevity. ■

The boundedness of $w_{ij}^k(t)$ can be established using arguments similar to Remark 1.

Remark 3. *In [23], a quantized adaptive pinning scheme was developed for synchronizing fractional-order multilayer networks. That work only adaptively adjusts the pinning strengths of selected nodes, while keeping the node coupling strengths and edge coupling weights constant. Consequently, achieving synchronization requires solving a linear matrix inequality that depends on the network topology and coupling parameters. In contrast, our approach adapts the intralayer coupling strengths of nodes or the coupling weights of edges. This allows any multiplex network with strongly connected intralayer topologies to achieve intralayer synchronization under the adaptive pinning strategy proposed here.*

IV. Numerical Examples

We consider a two-layer multiplex network, where each layer consists of four nodes. The intra-layer coupling matrices A^k, $k = 1, 2$, and the inter-layer coupling matrix B are given by

$$A^1 = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 1 & -2 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix},$$

$$A^2 = \begin{bmatrix} -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}.$$

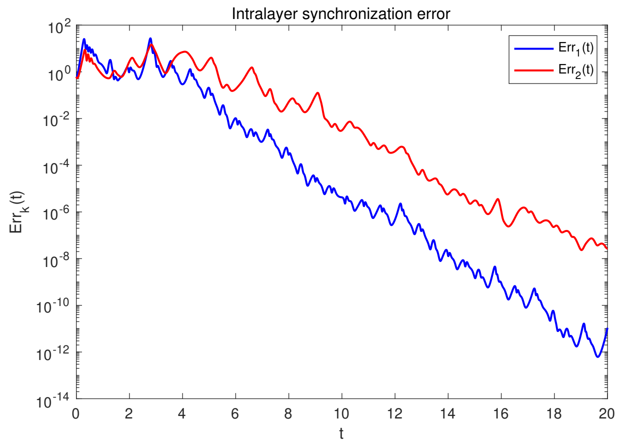

Fig. 1. The dynamics of intra-layer synchronization error $\mathrm{Err_k}(t), k=1,2$ with respect to t for the multiplex network (5) with adaptive coupling strengths (7) and adaptive pinning strengths (8).

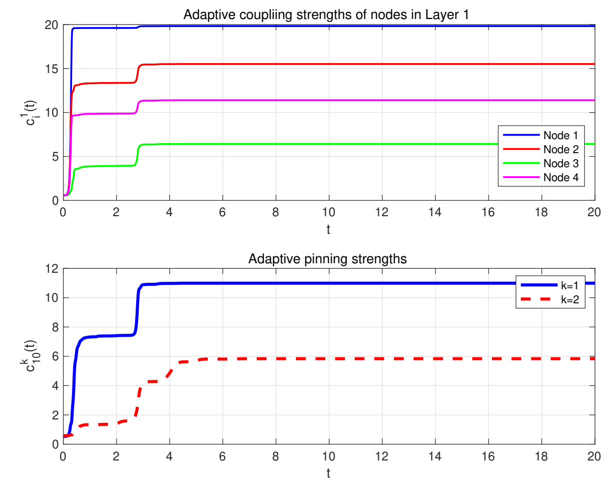

Fig. 2. Time evolution of coupling strengths $c_i^1(t)$ and pinning strengths $c_{40}^k(t), k = 1,2$ for the multiplex network (5) with adaptive coupling strengths (7) and adaptive pinning strengths (8).

For each layer, the fractional-order Liu system [24] is selected as the target trajectory,

$$\begin{aligned} {}_0^C D_t^\alpha y_1(t) &= a[y_2(t) - y_1(t)], \\ {}_0^C D_t^\alpha y_2(t) &= by_1(t) - ky_1(t)y_3(t), \\ {}_0^C D_t^\alpha y_3(t) &= -cy_3(t) + dy_1(t)^2. \end{aligned}$$

The parameters are chosen as

$$a = 10, b = 40, c = 2.5, d = 4, k = 1$$

for the first layer and

$$a = 10, b = 20, c = 2.8, d = 4, k = 2$$

for the second layer. Selecting $\alpha = 0.95$, the fourth node in each layer is chosen as the pinned node.

Define the synchronization error of the k-th layer ($k = 1,2$) as

$$Err_k(t) = \sum_{i=1}^{4} [x_i^k(t) - s_k(t)]^\top [x_i^k(t) - s_k(t)]$$

Consider the multiplex network (5) with adaptive node-based coupling strengths (7) and adaptive pinning strengths (8). The trajectories of the synchronization error $Err_k(t)$ are plotted in Fig. 1, demonstrating that intra-layer synchronization is achieved.

Fig. 2 plots the evolution of the coupling strengths $c_i^1(t)$ and pinning strengths $c_{i0}^k(t)$, showing that these adaptive parameters converge to positive steady values.

The multiplex network also achieves intralayer synchronization under the adaptive coupling weight update law (25). The corresponding simulation results, being qualitatively similar to those presented in Figs. 1 and 2, are omitted for conciseness.

V. Conclusions

This paper investigated pinning control of fractional-order multiplex networks via adaptive strategies. Fully distributed node-based and edge-based adaptive algorithms were proposed, and sufficient conditions for intra-layer synchronization were established using Lyapunov theory. Numerical simulations validated the effectiveness of the proposed methods.

Future work will focus on two main directions. First, extending the proposed adaptive algorithms to more general fractional-order multiplex networks with time delays and switching topologies. Second, incorporating event-triggered mechanisms into adaptive control frameworks to further reduce communication and computational costs in fractional-order multiplex networks.

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